

# OPENNESS STABILITY AND IMPLICIT MULTIFUNCTION THEOREMS. APPLICATIONS TO VARIATIONAL SYSTEMS

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**Abstract:** In this paper we aim to present two general results regarding, on one hand, the openness stability of set-valued maps and, on the other hand, the metric regularity behavior of the implicit multifunction related to a generalized variational system. Then, these results are applied in order to obtain, in a natural way, and in a widely studied case, several relations between the metric regularity moduli of the field maps defining the variational system and the solution map. Our approach allows us to complete and extend several very recent results in literature.

**Keywords:** set-valued mappings · linear openness · metric regularity · Lipschitz-like property · implicit multifunctions

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## 1 Introduction

This paper belongs to the active area of research concerning parametric variational systems and it aims to enter into dialog with some very recent works of Aragon Artacho and Mordukhovich ([1], [2]) and Ngai, Tron and Théra ([10]). Note that, in turn, these papers extend many results of Dontchev and Rockafellar ([3], [4]).

Firstly, our research on the relations between metric regularity/Lipschitz moduli of an initial parametric field map and the associated implicit multifunction map led us to the rediscovery of a very nice Theorem of Ursescu [12] concerning the stability of openness of set-valued maps (Theorem 3.1 below). The proof we provide here for this result is appropriate enough for getting some extra assertions compared with the initial paper of Ursescu. Later in the paper, this result is a key ingredient in order to get a natural and precise answer to the question of how regularity constants of the involved maps relate each other.

Secondly, we were interested in enlarging the framework commonly used as being the defining form of a parametric variational system to the case of a general field map. To be more specific, let

$X, Y, P$  be Banach spaces,  $H : X \times P \rightrightarrows Y$  be a multifunction and define the implicit set-valued map  $S : P \rightrightarrows X$  by

$$S(p) = \{x \in X \mid 0 \in H(x, p)\}.$$

Then, in this second part of our work we find how metric regularity and Lipschitz properties of  $H$  and  $S$  are related under certain assumptions (Theorem 3.6). This second main result is in fact a general implicit multifunction theorem. After fixing these two main tools we are able to present in a natural way the situation where  $H$  is a sum of two set-valued maps  $F, G$  of the form  $H(x, p) = F(x, p) + G(x)$ . Note that this case is more general than the situation considered in [1] by the presence of the set-valued map  $F$  instead of a single-valued map. Moreover, this is the most general situation one can consider because it is not possible to get good results concerning the Lipschitz properties when  $G$  depends on the parameter  $p$ , in virtue of [1, Remark 3.6. (iii)]. However, note that in ([10]) the authors deal with a sort of metric regularity of the solution map associated to the sum of two parametric set-valued maps. In our framework, when we put at work together the two main results, we are in the position to indicate in a smooth manner the relations between the regularity moduli of  $S, G$  and  $F$ . We hope that our main results and their combination will bring more light on the previous results on this topic.

We would like to mention that, in comparison with [1] and [2], we get here only results in which the assumptions are on  $F$  and  $G$  and the conclusion concerns  $S$ . The converse situation considered in the quoted works (from  $S$  to  $G$ ) is not presented here because it has too many similarities with the corresponding results of Artacho and Mordukhovich. Any interested reader could find the arguments to obtain such results in our framework, but with Artacho and Mordukhovich tools.

The paper is organized as follows. In the next section we present the notations, the concepts and the basic facts we use in the sequel. The third section contains the main results of the paper we have presented in few words above. The last section investigates the widely studied form of the parametric variational systems we can find in literature. We show here how the main results concerning the stability of the linear openness of set-valued maps and the implicit multifunctions could be combined in order to get quite easily the estimation of the regularity constants of the solution map.

## 2 Preliminaries

This section contains some basic definitions and results used in the sequel. In what follows, we suppose that all the involved spaces are Banach. In this setting,  $B(x, r)$  and  $D(x, r)$  denote the open and the closed ball with center  $x$  and radius  $r$ , respectively. Sometimes we write  $\mathbb{D}_X$  for the closed unit ball of  $X$ . If  $x \in X$  and  $A \subset X$ , one defines the distance from  $x$  to  $A$  as  $d(x, A) := \inf\{\|x - a\| \mid a \in A\}$ . As usual, we use the convention  $d(x, \emptyset) = \infty$ . For a non-empty set  $A \subset X$  we put  $\text{cl } A$ ,  $\text{int } A$  for the topological closure and interior, respectively. When we work on a product space, we consider the sum norm, unless otherwise stated.

Consider now a multifunction  $F : X \rightrightarrows Y$ . The domain and the graph of  $F$  are denoted respectively by

$$\text{Dom } F := \{x \in X \mid F(x) \neq \emptyset\}$$

and

$$\text{Gr } F = \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

If  $A \subset X$  then  $F(A) := \bigcup_{x \in A} F(x)$ . The inverse set-valued map of  $F$  is  $F^{-1} : Y \rightrightarrows X$  given by  $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$ .

Recall that a multifunction  $F$  is inner semicontinuous at  $(x, y) \in \text{Gr } F$  if for every open set  $D \subset Y$  with  $y \in D$ , there exists a neighborhood  $U \in \mathcal{V}(x)$  such that for every  $x' \in U$ ,  $F(x') \cap D \neq \emptyset$  (where  $\mathcal{V}(x)$  stands for the system of the neighborhoods of  $x$ ).

We remind now the concepts of openness at linear rate, metric regularity and Lipschitz-likeness of a multifunction around the reference point.

**Definition 2.1** Let  $L > 0$ ,  $F : X \rightrightarrows Y$  be a multifunction and  $(\bar{x}, \bar{y}) \in \text{Gr } F$ .

(i)  $F$  is said to be open at linear rate  $L > 0$ , or  $L$ -open around  $(\bar{x}, \bar{y})$  if there exist a positive number  $\varepsilon > 0$  and two neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{y})$  such that, for every  $\rho \in ]0, \varepsilon[$  and every  $(x, y) \in \text{Gr } F \cap [U \times V]$ ,

$$B(y, \rho L) \subset F(B(x, \rho)). \quad (2.1)$$

The supremum of  $L > 0$  over all the combinations  $(L, U, V, \varepsilon)$  for which (2.1) holds is denoted by  $\text{lop } F(\bar{x}, \bar{y})$  and is called the exact linear openness bound, or the exact covering bound of  $F$  around  $(\bar{x}, \bar{y})$ .

(ii)  $F$  is said to be Lipschitz-like, or has Aubin property around  $(\bar{x}, \bar{y})$  with constant  $L > 0$  if there exist two neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{y})$  such that, for every  $x, u \in U$ ,

$$F(x) \cap V \subset F(u) + L \|x - u\| \mathbb{D}_Y. \quad (2.2)$$

The infimum of  $L > 0$  over all the combinations  $(L, U, V)$  for which (2.2) holds is denoted by  $\text{lip } F(\bar{x}, \bar{y})$  and is called the exact Lipschitz bound of  $F$  around  $(\bar{x}, \bar{y})$ .

(iii)  $F$  is said to be metrically regular around  $(\bar{x}, \bar{y})$  with constant  $L > 0$  if there exist two neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{y})$  such that, for every  $(x, y) \in U \times V$ ,

$$d(x, F^{-1}(y)) \leq L d(y, F(x)). \quad (2.3)$$

The infimum of  $L > 0$  over all the combinations  $(L, U, V)$  for which (2.3) holds is denoted by  $\text{reg } F(\bar{x}, \bar{y})$  and is called the exact regularity bound of  $F$  around  $(\bar{x}, \bar{y})$ .

The next proposition contains the well-known links between the notions presented above. See [9, Theorems 1.49, 1.52] for more details about the proof.

**Proposition 2.2** Let  $F : X \rightrightarrows Y$  be a multifunction and  $(\bar{x}, \bar{y}) \in \text{Gr } F$ . Then  $F$  is open at linear rate around  $(\bar{x}, \bar{y})$  iff  $F^{-1}$  is Lipschitz-like around  $(\bar{y}, \bar{x})$  iff  $F$  is metrically regular around  $(\bar{x}, \bar{y})$ . Moreover, in every of the previous situations,

$$(\text{lop } F(\bar{x}, \bar{y}))^{-1} = \text{lip } F^{-1}(\bar{y}, \bar{x}) = \text{reg } F(\bar{x}, \bar{y}).$$

It is well known that the corresponding "at point" properties are significantly different from the "around point" ones. Let us introduce now some of these notions. For more related concepts we refer to [1].

**Definition 2.3** Let  $L > 0$ ,  $F : X \rightrightarrows Y$  be a multifunction and  $(\bar{x}, \bar{y}) \in \text{Gr } F$ .

(i)  $F$  is said to be open at linear rate  $L$ , or  $L$ -open at  $(\bar{x}, \bar{y})$  if there exists a positive number  $\varepsilon > 0$  such that, for every  $\rho \in ]0, \varepsilon[$ ,

$$B(\bar{y}, \rho L) \subset F(B(\bar{x}, \rho)). \quad (2.4)$$

The supremum of  $L > 0$  over all the combinations  $(L, \varepsilon)$  for which (2.4) holds is denoted by  $\text{plop } F(\bar{x}, \bar{y})$  and is called the exact punctual linear openness bound of  $F$  at  $(\bar{x}, \bar{y})$ .

(ii)  $F$  is said to be pseudocalm with constant  $L$ , or  $L$ -pseudocalm at  $(\bar{x}, \bar{y})$ , if there exists a neighborhood  $U \in \mathcal{V}(\bar{x})$  such that, for every  $x \in U$ ,

$$d(\bar{y}, F(x)) \leq L \|x - \bar{x}\|. \quad (2.5)$$

The infimum of  $L > 0$  over all the combinations  $(L, U)$  for which (2.5) holds is denoted by  $\text{psdclm } F(\bar{x}, \bar{y})$  and is called the exact bound of pseudocalmness for  $F$  at  $(\bar{x}, \bar{y})$ .

(iii)  $F$  is said to be metrically hemiregular with constant  $L$ , or  $L$ -metrically hemiregular at  $(\bar{x}, \bar{y})$  if there exists a neighborhood  $V \in \mathcal{V}(\bar{y})$  such that, for every  $y \in V$ ,

$$d(\bar{x}, F^{-1}(y)) \leq L \|y - \bar{y}\|. \quad (2.6)$$

The infimum of  $L > 0$  over all the combinations  $(L, V)$  for which (2.6) holds is denoted by  $\text{hemreg } F(\bar{x}, \bar{y})$  and is called the exact hemiregularity bound of  $F$  at  $(\bar{x}, \bar{y})$ .

The term of metric hemiregularity appears in [2, Definition 5.1], where the link with "Lipschitz lower semicontinuity" (i.e., pseudocalmness in our terminology) of the inverse multifunction is emphasized. The notion of pseudocalmness is used under the term of  $L$ -Lipschitz in [12], where other concepts of relative openness and relative  $L$ -Lipschitz properties are introduced and discussed.

The next proposition lists some equivalences between these "at point" notions. We give the (elementary) proof for the completeness.

**Proposition 2.4** *Let  $L > 0$ ,  $F : X \rightrightarrows Y$  and  $(\bar{x}, \bar{y}) \in \text{Gr } F$ . Then  $F$  is  $L$ -open at  $(\bar{x}, \bar{y})$  iff  $F^{-1}$  is  $L^{-1}$ -pseudocalm at  $(\bar{y}, \bar{x})$  iff  $F$  is  $L^{-1}$ -metrically hemiregular at  $(\bar{x}, \bar{y})$ . Moreover, in every of the previous situations,*

$$(\text{plop } F(\bar{x}, \bar{y}))^{-1} = \text{psdclm } F^{-1}(\bar{y}, \bar{x}) = \text{hemreg } F(\bar{x}, \bar{y}).$$

**Proof.** It's obvious from the very definitions that  $F^{-1}$  is  $L^{-1}$ -pseudocalm at  $(\bar{y}, \bar{x})$  iff  $F$  is  $L^{-1}$ -metrically hemiregular at  $(\bar{x}, \bar{y})$ . Suppose now that  $F$  is  $L$ -open at  $(\bar{x}, \bar{y})$ . Then there exists  $\varepsilon > 0$  such that, for every  $\rho \in ]0, \varepsilon[$ , (2.4) holds. Consider  $\varepsilon' := L\varepsilon$  and take arbitrarily  $y \in B(\bar{y}, \varepsilon')$ . Then there exist  $\rho \in ]0, \varepsilon[$  and  $\gamma$  arbitrary small such that  $\|y - \bar{y}\| = L\rho < L(\rho + \gamma) < L\varepsilon$ . Using the  $L$ -openness of  $F$  at  $(\bar{x}, \bar{y})$ ,  $y \in B(\bar{y}, L(\rho + \gamma)) \subset F(B(\bar{x}, \rho + \gamma))$ . Consequently, one can find  $x \in B(\bar{x}, \rho + \gamma) \cap F^{-1}(y)$ . Then, for  $\gamma$  arbitrary small,  $d(\bar{x}, F^{-1}(y)) \leq \|x - \bar{x}\| < \rho + \gamma$ , whence  $d(\bar{x}, F^{-1}(y)) \leq \rho = L^{-1} \|y - \bar{y}\|$ , and the first implication is now proved.

Suppose now that  $F^{-1}$  is  $L^{-1}$ -pseudocalm at  $(\bar{y}, \bar{x})$ , so there exists  $\varepsilon > 0$  such that, for every  $y \in B(\bar{y}, \varepsilon)$ ,  $d(\bar{x}, F^{-1}(y)) \leq L^{-1} \|y - \bar{y}\|$ . Take  $\varepsilon' := L^{-1}\varepsilon$  and arbitrarily  $\rho \in ]0, \varepsilon'[$ . If  $y' \in B(\bar{y}, L\rho)$ , we obtain that  $\|y' - \bar{y}\| < L\varepsilon' = \varepsilon$ , so  $d(\bar{x}, F^{-1}(y')) \leq L^{-1} \|y' - \bar{y}\| < \rho$ . Consequently, there exists  $x' \in F^{-1}(y')$  such that  $\|x' - \bar{x}\| < \rho$ , whence  $y' \in F(x') \subset F(B(\bar{x}, \rho))$ . The proof is now complete.  $\square$

See [13, Section 11] for an example of a multifunction which is open at linear rate at a point, hence on the basis of Proposition 2.4 is metrically hemiregular at this point, even if that multifunction is not metrically regular around any point.

Recall that  $\mathcal{L}(X, Y)$  denotes the normed vector space of linear bounded operators acting between  $X$  and  $Y$ . If  $A \in \mathcal{L}(X, Y)$ , then the "at" and "around point" notions do coincide. In fact,  $A$  is metrically regular around every  $x \in X$  iff  $A$  is metrically hemiregular at every  $x \in X$  iff  $A$  is open

with linear rate around every  $x \in X$  iff  $A$  is open with linear rate at every  $x \in X$  iff  $A$  is surjective. Moreover, in every of these cases we have

$$\text{hemreg } A = \text{reg } A = (\text{plop } A)^{-1} = (\text{lop } A)^{-1} = \|(A^*)^{-1}\|,$$

where  $A^* \in \mathcal{L}(Y^*, X^*)$  denotes the adjoint operator and  $\text{hemreg } A$ ,  $\text{reg } A$ ,  $\text{plop } A$  and  $\text{lop } A$  are common for all the points  $x \in X$  (see, for more details, [2, Proposition 5.2]).

In the following, we introduce the corresponding partial notions of linear openness, metric regularity and Lipschitz-like property around the reference point for a parametric set-valued map. Below, we denote by  $P$  the Banach space of parameters.

**Definition 2.5** *Let  $L > 0$ ,  $F : X \times P \rightrightarrows Y$  be a multifunction,  $((\bar{x}, \bar{p}), \bar{y}) \in \text{Gr } F$  and for every  $p \in P$ , denote  $F_p(\cdot) := F(\cdot, p)$ .*

*(i)  $F$  is said to be open at linear rate  $L > 0$ , or  $L$ -open, with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), \bar{y})$  if there exist a positive number  $\varepsilon > 0$  and some neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{p})$ ,  $W \in \mathcal{V}(\bar{y})$  such that, for every  $\rho \in ]0, \varepsilon[$ , every  $p \in V$  and every  $(x, y) \in \text{Gr } F_p \cap [U \times W]$ ,*

$$B(y, \rho L) \subset F_p(B(x, \rho)), \quad (2.7)$$

*The supremum of  $L > 0$  over all the combinations  $(L, U, V, W, \varepsilon)$  for which (2.7) holds is denoted by  $\widehat{\text{lop}}_x F((\bar{x}, \bar{p}), \bar{y})$  and is called the exact linear openness bound, or the exact covering bound of  $F$  in  $x$  around  $((\bar{x}, \bar{p}), \bar{y})$ .*

*(ii)  $F$  is said to be Lipschitz-like, or has Aubin property, with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), \bar{y})$  with constant  $L > 0$  if there exist some neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{p})$ ,  $W \in \mathcal{V}(\bar{y})$  such that, for every  $x, u \in U$  and every  $p \in V$ ,*

$$F_p(x) \cap W \subset F_p(u) + L \|x - u\| \mathbb{D}_Y. \quad (2.8)$$

*The infimum of  $L > 0$  over all the combinations  $(L, U, V, W)$  for which (2.8) holds is denoted by  $\widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y})$  and is called the exact Lipschitz bound of  $F$  in  $x$  around  $((\bar{x}, \bar{p}), \bar{y})$ .*

*(iii)  $F$  is said to be metrically regular with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), \bar{y})$  with constant  $L > 0$  if there exist some neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{p})$ ,  $W \in \mathcal{V}(\bar{y})$  such that, for every  $(x, p, y) \in U \times V \times W$ ,*

$$d(x, F_p^{-1}(y)) \leq L d(y, F_p(x)). \quad (2.9)$$

*The infimum of  $L > 0$  over all the combinations  $(L, U, V, W)$  for which (2.9) holds is denoted by  $\widehat{\text{reg}}_x F((\bar{x}, \bar{p}), \bar{y})$  and is called the exact regularity bound of  $F$  in  $x$  around  $((\bar{x}, \bar{p}), \bar{y})$ .*

Similarly, one can define the notions of linear openness, metric regularity and Lipschitz-like property with respect to  $p$  uniformly in  $x$ , and the corresponding exact bounds.

### 3 Main results

We begin our analysis with an interesting result due to Ursescu (see, e.g., [12, Theorem 1]), which brings into the light the key fact that the linear openness property of a difference type multifunction can be deduced from the corresponding linear openness properties of its terms. This result can be viewed as a deep generalization of the Graves Theorem (see the remark after Theorem 3.1) and is, to the best of our knowledge, the most general assertion of this type existing in literature.

Moreover, its importance seems to be crucial, and maybe underevaluated by now, although it can be putted into relation with a large number of (very) actual topics, as the strongly regular generalized equations of Robinson type (see [11]), allowing for the first time to deal with problems where both the terms are multivalued, but also with the inverse and implicit type theorems for set-valued mappings, with parametric variational inclusions and more, as one can see next. Nevertheless, its original proof is rather complicated and this is the place where we would like to add a small contribution. Namely, we use a method of proof inaugurated in early '80s in some papers of Penot and Ioffe and subsequently constantly used in openness results. In this line, we exploit an idea already used in [5], we apply the Ekeland Variational Principle in a slightly different way (on the cartesian product), and the proof of Ursescu's result becomes much more natural and simple. Besides the result itself, this new proof of it is a key element in extending the framework to the parametric case and, furthermore, to the rest of the applications we make precise in the last section.

**Theorem 3.1** *Let  $F : X \rightrightarrows Y$  and  $G : Y \rightrightarrows X$  be two multifunctions such that  $\text{Gr } F$  and  $\text{Gr } G$  are locally closed. Suppose that  $\text{Dom}(F - G^{-1})$  and  $\text{Dom}(G - F^{-1})$  are nonempty and let  $L > 0$  and  $M > 0$  be such that  $LM > 1$ . If  $F$  is  $L$ -open at every point of its graph, and  $G$  is  $M$ -open at every point of its graph, then  $F - G^{-1}$  is  $(L - M^{-1})$ -open at every point of its graph and  $G - F^{-1}$  is  $(M - L^{-1})$ -open at every point of its graph.*

**Proof.** We prove only the first assertion, the other one being completely symmetrical. Let  $(x, w) \in \text{Gr}(F - G^{-1})$ . Then there exist  $y \in F(x)$  and  $z \in G^{-1}(x)$  such that  $w = y - z$ . Define the multifunction  $(F, G^{-1}) : X \rightrightarrows Y \times Y$  by  $(F, G^{-1})(x) := F(x) \times G^{-1}(x)$  and remark that  $(x, y, z) \in \text{Gr}(F, G^{-1})$ . Because  $\text{Gr } F$  and  $\text{Gr } G$  are locally closed, it follows that  $\text{Gr}(F, G^{-1})$  is locally closed and one can find  $\rho > 0$  such that  $\text{Gr}(F, G^{-1}) \cap \text{cl } W$  is closed, where

$$W := B(x, \rho) \times B(y, L\rho) \times B(z, M^{-1}\rho). \quad (3.1)$$

Take  $u \in B(w, (L - M^{-1})\rho)$ . We must prove that  $u \in (F - G^{-1})(B(x, \rho))$ . One can find  $\tau \in ]0, 1[$  such that  $\|u - w\| < \tau(L - M^{-1})\rho$ . Endow the space  $X \times Y \times Y$  with the norm

$$\|(p, q, r)\|_0 := \tau(L - M^{-1}) \max\{\|p\|, L^{-1}\|q\|, M\|r\|\}$$

and apply the Ekeland variational principle to the function  $h : \text{Gr}(F, G^{-1}) \cap \text{cl } W \rightarrow \mathbb{R}_+$ ,

$$h(p, q, r) := \|u - (q - r)\|.$$

Then one can find a point  $(a, b, c) \in \text{Gr}(F, G^{-1}) \cap \text{cl } W$  such that

$$\|u - (b - c)\| \leq \|u - (y - z)\| - \|(a, b, c) - (x, y, z)\|_0 \quad (3.2)$$

and

$$\|u - (b - c)\| \leq \|u - (q - r)\| + \|(a, b, c) - (p, q, r)\|_0, \text{ for every } (p, q, r) \in \text{Gr}(F, G^{-1}) \cap \text{cl } W. \quad (3.3)$$

From (3.2) we have that

$$\begin{aligned} \tau(L - M^{-1}) \max\{\|a - x\|, L^{-1}\|b - y\|, M\|c - z\|\} &= \|(a, b, c) - (x, y, z)\|_0 \\ &\leq \|u - (y - z)\| = \|u - w\| \\ &< \tau(L - M^{-1})\rho, \end{aligned}$$

hence  $(a, b, c) \in W$ , and, in particular,  $a \in B(x, \rho)$ .

If  $u = b - c$ , then  $u \in (F - G^{-1})(a) \subset (F - G^{-1})(B(x, \rho))$  and the desired assertion is proved.

We want to show that  $u = b - c$  is the sole possible situation. For this, suppose by means of contradiction that  $u \neq b - c$ . Fix  $\varepsilon > 0$  such that  $L - \varepsilon > 0$  and define next

$$v := (L - \varepsilon) \|u - (b - c)\|^{-1} (u - (b - c)).$$

Then, for every  $s > 0$  sufficiently small, from the  $L$ -openness of  $F$  at  $(a, b) \in \text{Gr } F$ , we obtain that

$$b + sv \in B(b, Ls) \subset F(B(a, s)).$$

Consequently, there exists  $d \in B(a, s)$  such that  $b + sv \in F(d)$ . Obviously, one can find  $p$  with  $\|p\| < 1$  such that  $d = a + sp$ .

Also, using the  $M$ -openness of  $G$  at  $(c, a)$ , we have that for every  $s > 0$  sufficiently small,

$$B(a, s) \subset G(B(c, M^{-1}s)).$$

Hence, one can find  $e \in B(c, M^{-1}s)$  such that  $d \in G(e)$  or, equivalently,  $e \in G^{-1}(d)$ . Because we can write  $e = c + M^{-1}sq$  with  $\|q\| < 1$ , we finally have that  $(a + sp, b + sv, c + M^{-1}sq) \in \text{Gr}(F, G^{-1})$ . Taking a smaller  $s$  if necessary, we also have that  $(a + sp, b + sv, c + M^{-1}sq) \in W$ . We use now (3.3) to obtain that, for every  $s > 0$  sufficiently small,

$$\begin{aligned} \|u - (b - c)\| &\leq \|u - (b + sv - c - M^{-1}sq)\| + s \|(p, v, M^{-1}q)\|_0 \\ &\leq \|u - (b - c) - sv\| + M^{-1}s + s \|(p, v, M^{-1}q)\|_0. \end{aligned} \quad (3.4)$$

But

$$\|u - (b - c) - sv\| = \|\|u - (b - c)\| - s(L - \varepsilon)\|.$$

Eventually for even a smaller  $s$ , one obtains successively from (3.4) that

$$\begin{aligned} \|u - (b - c)\| &\leq \|u - (b - c)\| - s(L - \varepsilon) + M^{-1}s + s \|(p, v, M^{-1}q)\|_0, \\ L - M^{-1} - \varepsilon &\leq \tau(L - M^{-1}) \max\{\|p\|, L^{-1}\|v\|, M\|M^{-1}q\|\}, \\ L - M^{-1} - \varepsilon &\leq \tau(L - M^{-1}). \end{aligned}$$

Passing to the limit when  $\varepsilon \rightarrow 0$ , we get that  $1 \leq \tau$ , which is the contradiction. The proof is now complete.  $\square$

Let us point out that the previous theorem contains several results in literature, especially when  $F$  and  $G$  are single-valued (see, for more details, [12, p. 412]). Among these results, maybe the most famous one is that of Graves [8, p. 112], which can be easily deduced from Theorem 3.1 and Proposition 2.4. See, also, the comment after Corollary 3.4.

Notice that with a slight modification of the proof of Theorem 3.1, one obtains the next straightforward generalization (see [12, Theorem 3]).

**Theorem 3.2** *Let  $F : X \rightrightarrows Y$  and  $G_1, \dots, G_n : Y \rightrightarrows X$  be such that  $\text{Gr } F$  and  $\text{Gr } G_i$ ,  $i \in \{1, 2, \dots, n\}$ , are locally closed. Suppose  $\text{Dom}(F - G_1^{-1} - \dots - G_n^{-1})$  is nonempty and let  $L > 0$  and  $M_1, \dots, M_n > 0$  be such that  $L > M_1^{-1} + \dots + M_n^{-1}$ . If  $F$  is  $L$ -open at every point from a neighborhood of  $(x, y) \in \text{Gr } F$ , and  $G_i$  is  $M_i$ -open at every point from a neighborhood of  $(z_i, x) \in \text{Gr } G_i$  for every  $i \in \{1, 2, \dots, n\}$ , then  $F - G_1^{-1} - \dots - G_n^{-1}$  is  $(L - M_1^{-1} - \dots - M_n^{-1})$ -open at  $(x, y - z_1 - \dots - z_n)$ .*



**Proof.** Proceed as above, by taking the multifunction  $(F, G_1^{-1}, \dots, G_n^{-1}) : X \rightrightarrows Y^{n+1}$  given as

$$(F, G_1^{-1}, \dots, G_n^{-1})(x) := F(x) \times G_1^{-1}(x) \times \dots \times G_n^{-1}(x)$$

and observe that  $(x, y, z_1, \dots, z_n) \in \text{Gr}(F, G_1^{-1}, \dots, G_n^{-1})$ . The rest of the proof is more or less identical with that of Theorem 3.1, using now

$$W := B(x, \rho) \times B(y, L\rho) \times B(z_1, M_1^{-1}\rho) \times \dots \times B(z_n, M_n^{-1}\rho)$$

such that  $\text{Gr}(F, G_1^{-1}, \dots, G_n^{-1}) \cap \text{cl } W$  is closed, endowing the space  $X \times Y^{n+1}$  with the norm

$$\|(p, q, r_1, \dots, r_n)\|_0 := \tau(L - M_1^{-1} - \dots - M_n^{-1}) \max\{\|p\|, L^{-1}\|q\|, M_1\|r_1\|, \dots, M_n\|r_n\|\}$$

and applying the Ekeland variational principle to the function  $h : \text{Gr}(F, G_1^{-1}, \dots, G_n^{-1}) \cap \text{cl } W \rightarrow \mathbb{R}_+$ ,

$$h(p, q, r_1, \dots, r_n) := \|u - (q - r_1 - \dots - r_n)\|. \quad \square$$

The role of the next theorem is to precisely specify the constants involved in Theorem 3.1. This will be a key ingredient in the proof of several subsequent results.

**Theorem 3.3** *Let  $F : X \rightrightarrows Y$  and  $G : Y \rightrightarrows X$  be two multifunctions and  $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Y$  such that  $(\bar{x}, \bar{y}) \in \text{Gr } F$  and  $(\bar{z}, \bar{x}) \in \text{Gr } G$ . Suppose that the following assumptions are satisfied:*

- (i)  *$\text{Gr } F$  is locally closed around  $(\bar{x}, \bar{y})$ , so there exist  $\alpha_1, \beta_1 > 0$  such that  $\text{Gr } F \cap \text{cl}[B(\bar{x}, \alpha_1) \times B(\bar{y}, \beta_1)]$  is closed;*
- (ii)  *$\text{Gr } G$  is locally closed around  $(\bar{z}, \bar{x})$ , so there exist  $\alpha_2, \beta_2 > 0$  such that  $\text{Gr } G \cap \text{cl}[B(\bar{z}, \beta_2) \times B(\bar{x}, \alpha_2)]$  is closed;*
- (iii) *there exist  $L, r_1, s_1 > 0$  such that, for every  $(x', y') \in \text{Gr } F \cap [B(\bar{x}, r_1) \times B(\bar{y}, s_1)]$ ,  $F$  is  $L$ -open at  $(x', y')$ ;*
- (iv) *there exist  $M, r_2, s_2 > 0$  such that, for every  $(v', u') \in \text{Gr } G \cap [B(\bar{z}, s_2) \times B(\bar{x}, r_2)]$ ,  $G$  is  $M$ -open at  $(v', u')$ ;*
- (v)  *$LM > 1$ .*

*Then for every  $\rho \in ]0, \varepsilon[$ , where  $\varepsilon := \min\{\alpha_1, \alpha_2, L^{-1}\beta_1, M\beta_2, r_1, r_2, L^{-1}s_1, Ms_2\}$ ,*

$$B(\bar{y} - \bar{z}, (L - M^{-1})\rho) \subset (F - G^{-1})(B(\bar{x}, \rho)).$$

*Moreover, for every  $\rho \in (0, 2^{-1}\varepsilon)$  and every  $(x, y, z) \in \text{Gr}(F, G^{-1}) \cap [B(\bar{x}, 2^{-1}\varepsilon) \times B(\bar{y}, 2^{-1}\varepsilon) \times B(\bar{z}, 2^{-1}\varepsilon)]$ ,*

$$B(y - z, (L - M^{-1})\rho) \subset (F - G^{-1})(B(x, \rho)).$$

**Proof.** We only sketch the proof, pointing out the differences with respect to the proof of Theorem 3.1.

For the first part, take  $\rho \in ]0, \varepsilon[$ , define, as above, the multifunction  $(F, G^{-1})$ , and observe that the choice of  $\varepsilon$  implies that  $\text{Gr}(F, G^{-1}) \cap \text{cl } W$  is closed, where  $W := B(\bar{x}, \rho) \times B(\bar{y}, L\rho) \times B(\bar{z}, M^{-1}\rho)$ .

Take again  $u \in B(\bar{y} - \bar{z}, (L - M^{-1})\rho)$  and follow the same steps as above to obtain that  $(a, b, c) \in W$ . We only need to know that  $F$  is  $L$ -open at  $(a, b)$  and that  $G$  is  $M$ -open at  $(c, a)$  to complete the proof, but this follows again from the choice of  $\varepsilon$ .

For the second part, we define  $W_1 := B(x, \rho) \times B(y, L\rho) \times B(z, M^{-1}\rho)$  and we remark again that  $\text{Gr}(F, G^{-1}) \cap \text{cl } W_1$  is closed, because  $W_1 \subset B(\bar{x}, \min\{\alpha_1, \alpha_2\}) \times B(\bar{y}, \beta_1) \times B(\bar{z}, \beta_2)$ . The rest



of the proof is the same as above, observing only that because  $(a, b, c) \in W_1 \subset B(\bar{x}, \min\{r_1, r_2\}) \times B(\bar{y}, s_1) \times B(\bar{z}, s_2)$ ,  $F$  is  $L$ -open at  $(a, b)$  and  $G$  is  $M$ -open at  $(c, a)$ .  $\square$

We want to emphasize that if  $F$  is  $L$ -open around  $(x, y)$ , then  $F$  satisfies the property from the third item, and a similar observation is valid for  $G$ . Also, if one of the two multifunctions which appear in the previous result is univoque, then one can obtain the openness around the reference point of the difference.

**Corollary 3.4** *Let  $f : X \rightarrow Y$  be a function,  $G : Y \rightrightarrows X$  be a multifunction,  $L, M > 0$  and  $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Y$  such that  $\bar{y} = f(\bar{x})$  and  $(\bar{z}, \bar{x}) \in \text{Gr } G$ . Suppose that the following assumptions are satisfied:*

- (i)  $f$  is Lipschitz continuous around  $\bar{x}$ ;
  - (ii)  $\text{Gr } G$  is locally closed around  $(\bar{z}, \bar{x})$ ;
  - (iii)  $f$  is  $L$ -open around  $(\bar{x}, \bar{y})$ ;
  - (iv)  $G$  is  $M$ -open around  $(\bar{z}, \bar{x})$ ;
  - (v)  $LM > 1$ .
- Then  $f - G^{-1}$  is  $(L - M^{-1})$ -open around  $(\bar{x}, \bar{y} - \bar{z})$ .*

**Proof.** From the Lipschitz property, we obtain the local closedness of  $\text{Gr } f$  around  $(\bar{x}, \bar{y})$ , so one deduces a similar assertion as in (i) of the previous Corollary. Observe also that the rest of the assumptions are the same or stronger compared to the previous result, so we suppose in the following that the conditions are formulated using the same constants as above. Define again  $\varepsilon := \min\{\alpha_1, \alpha_2, L^{-1}\beta_1, M\beta_2, r_1, r_2, L^{-1}s_1, Ms_2\}$ ,  $\gamma := \min\{4^{-1}\varepsilon, (4l)^{-1}\varepsilon\}$ , where  $l > 0$  is the Lipschitz constant from (i). Take now  $\rho \in ]0, 2^{-1}\varepsilon[$  and  $(x, v) \in \text{Gr}(f - G^{-1}) \cap [B(\bar{x}, \gamma) \times B(\bar{y} - \bar{z}, \gamma)]$ , so there exist  $y = f(x)$  and  $z \in G(x)$  such that  $v = y - z$ . Then we get that

$$\begin{aligned} x &\in B(\bar{x}, \gamma) \subset B(\bar{x}, 2^{-1}\varepsilon), \\ y &\in B(\bar{y}, l\gamma) \subset B(\bar{y}, 4^{-1}\varepsilon) \subset B(\bar{y}, 2^{-1}\varepsilon), \\ z &\in \bar{z} + (y - \bar{y}) + B(0, \gamma) \subset \bar{z} + B(0, 4^{-1}\varepsilon) + B(0, 4^{-1}\varepsilon) \subset B(\bar{z}, 2^{-1}\varepsilon), \end{aligned}$$

where we used (i) in the second row of inclusion.

Consequently,  $(x, y, z) \in \text{Gr}(f, G^{-1}) \cap [B(\bar{x}, 2^{-1}\varepsilon) \times B(\bar{y}, 2^{-1}\varepsilon) \times B(\bar{z}, 2^{-1}\varepsilon)]$ , so using the final part of the previous result we know that  $B(y - z, (L - M^{-1})\rho) \subset (f - G^{-1})(B(x, \rho))$ , and the proof is complete.  $\square$

As an easy consequence of the previous corollary, one can obtain the celebrated result of Lyusternik-Graves [8, p. 112], taking  $f$  as the Fréchet differential at a point of a continuously Fréchet differentiable function  $h$  and  $G := (h - f)^{-1}$ .

The next corollary, which can be seen as a parametric version of a result of Graves, is the same with [2, Proposition 3.2] and gives sufficient conditions for the partial metric regularity of a function, but here is obtained as an easy consequence of Corollary 3.4.

**Corollary 3.5** *Let  $f : X \times P \rightarrow Y$  be a function between Banach spaces which is continuous around  $(\bar{x}, \bar{p})$  and let  $A \in \mathcal{L}(P, Y)$  be a surjective linear operator such that there exists  $\alpha > 0$  such that  $\text{lop } A > \alpha$  and for every  $p, p'$  in a neighborhood  $U$  of  $\bar{p}$  and every  $x$  in a neighborhood  $V$  of  $\bar{x}$ ,*

$$\|f(x, p) - f(x, p') - A(p - p')\| \leq \alpha \|p - p'\|.$$

Then  $f$  is open with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), f(\bar{x}, \bar{p}))$  with

$$\widehat{\text{lop}}_p f(\bar{x}, \bar{p}) \geq \text{lop } A - \alpha.$$

Equivalently,  $f$  is metrically regular with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), f(\bar{x}, \bar{p}))$  with

$$\widehat{\text{reg}}_p f(\bar{x}, \bar{p}) \leq \frac{\text{reg } A}{1 - \alpha \cdot \text{reg } A}.$$

**Proof.** Just take  $x \in V$  and apply Corollary 3.4 for  $f := -A$  and  $G := (f(x, \cdot) - A)^{-1}$ .  $\square$

Note first that the constants in right-hand sides of the relations from the conclusion are coming directly from the openness result, as an easy consequence. Another remark concerns the fact that in the case where  $f$  is (strictly) partially differentiable with respect to  $x$ , then, as it is often the case in literature, one can take the partial differential with respect to  $x$  instead of  $A$  (see [11], [1]).

Here comes the second main result of the paper. For this, we use some ideas we have previously developed in [6]. Let us introduce the objects we deal with. Remind that, for a multifunction  $H : X \times P \rightrightarrows Y$ , we can define the implicit set-valued map  $S : P \rightrightarrows X$  by:

$$S(p) = \{x \in X \mid 0 \in H(x, p)\}.$$

Note that a more general solution map (see [6], [10]) could be investigated from the point of view of several metric regularity concepts. We prefer the use of  $S$  in the present form for clarity and unity of the results.

The result we present is an implicit multifunction theorem and shows some interesting interrelations between the partial openness with respect to a variable plus the Lipschitz-like property with respect to the other variable of the original multifunction, and the Lipschitz-like, or the metric regularity of the implicit multifunction, respectively.

**Theorem 3.6** *Let  $X, Y, P$  be Banach spaces,  $H : X \times P \rightrightarrows Y$  be a set-valued map and  $(\bar{x}, \bar{p}, 0) \in \text{Gr } H$ . Denote by  $H_p(\cdot) := H(\cdot, p)$ ,  $H_x(\cdot) := H(x, \cdot)$ , and suppose that  $H$  is inner semicontinuous at  $(\bar{x}, \bar{p}, 0)$ .*

*(i) If  $H$  is open with linear rate  $c > 0$  with respect to  $x$  uniformly in  $p$  around  $(\bar{x}, \bar{p}, 0)$ , then there exist  $\bar{r}, \bar{t} > 0$  such that, for every  $(x, p) \in B(\bar{x}, \bar{r}) \times B(\bar{p}, \bar{t})$ ,*

$$d(x, S(p)) \leq c^{-1} d(0, H(x, p)). \quad (3.5)$$

*If, moreover,  $H$  is Lipschitz-like with respect to  $p$  uniformly in  $x$  around  $(\bar{x}, \bar{p}, 0)$ , then  $S$  is Lipschitz-like around  $(\bar{p}, \bar{x})$  and*

$$\text{lip } S(\bar{p}, \bar{x}) \leq c^{-1} \widehat{\text{lip}}_p H((\bar{x}, \bar{p}), 0). \quad (3.6)$$

*(ii) If  $H$  is open with linear rate  $c > 0$  with respect to  $p$  uniformly in  $x$  around  $(\bar{x}, \bar{p}, 0)$ , then there exist  $\bar{r}, \bar{t} > 0$  such that, for every  $(x, p) \in B(\bar{x}, \bar{r}) \times B(\bar{p}, \bar{t})$ ,*

$$d(p, S^{-1}(x)) \leq c^{-1} d(0, H(x, p)). \quad (3.7)$$

*If, moreover,  $H$  is Lipschitz-like with respect to  $x$  uniformly in  $p$  around  $(\bar{x}, \bar{p}, 0)$ , then  $S$  is metrically regular around  $(\bar{p}, \bar{x})$  and*

$$\text{reg } S(\bar{p}, \bar{x}) \leq c^{-1} \widehat{\text{lip}}_x H((\bar{x}, \bar{p}), 0). \quad (3.8)$$

**Proof.** We will prove only the first item, because for the second one it suffices to observe that, defining the multifunction  $T := S^{-1}$ , the proof is completely symmetrical, using  $T$  instead of  $S$ . Moreover, using Proposition 2.2, we know that  $\text{reg } S(\bar{p}, \bar{x}) = \text{lip } T(\bar{x}, \bar{p})$  and then (3.8) follows from (3.6).

For the (i) item, we know that there exist  $r, s, t, c, \varepsilon > 0$  such that, for every  $\rho \in (0, \varepsilon)$ , every  $p \in B(\bar{p}, t)$  and every  $(x, y) \in \text{Gr } H_p \cap [B(\bar{x}, r) \times B(0, s)]$ ,

$$B(y, c\rho) \subset H_p(B(x, \rho)).$$

Take now  $\rho \in ]0, \min\{\varepsilon, c^{-1}s\}[$ . Because  $H$  is inner semicontinuous at  $(\bar{x}, \bar{p}, 0)$ , one can find  $\gamma, \delta > 0$  such that, for every  $(x, p) \in B(\bar{x}, \gamma) \times B(\bar{p}, \delta)$ ,

$$H(x, p) \cap B(0, c\rho) \neq \emptyset. \quad (3.9)$$

Choose  $\bar{r} := \min\{r, \gamma\}$ ,  $\bar{t} := \min\{t, \delta\}$  and take  $(x, p) \in B(\bar{x}, \bar{r}) \times B(\bar{p}, \bar{t})$ . If  $0 \in H(x, p)$ , then (3.5) trivially holds. Suppose now that  $0 \notin H(x, p)$ . Then for every  $\xi > 0$ , there exists  $y_\xi \in H(x, p)$  such that

$$\|y_\xi\| < d(0, H(x, p)) + \xi.$$

By the use of (3.9), we know that  $d(0, H(x, p)) < c\rho$ , whence we can choose  $\xi$  sufficiently small such that  $d(0, H(x, p)) + \xi < c\rho$ . Consequently,

$$0 \in B(y_\xi, d(0, H(x, p)) + \xi) \subset B(y_\xi, c\rho). \quad (3.10)$$

Observe now that  $x \in B(\bar{x}, r)$ ,  $p \in B(\bar{p}, t)$ ,  $y_\xi \in B(0, d(0, H(x, p)) + \xi) \subset B(0, c\rho) \subset B(0, s)$ ,  $y_\xi \in H(x, p)$  and denote  $\rho_0 := c^{-1}(d(0, H(x, p)) + \xi) < \rho < \varepsilon$ .

But we know that

$$B(y_\xi, c\rho_0) \subset H_p(B(x, \rho_0)),$$

hence, using also (3.10), one obtains that there exists  $x_0 \in B(x, \rho_0)$  such that  $0 \in H(x_0, p)$ , which is equivalent to  $x_0 \in S(p)$ . Then

$$d(x, S(p)) \leq \|x - x_0\| < \rho_0 = c^{-1}(d(0, H(x, p)) + \xi).$$

Making  $\xi \rightarrow 0$ , we obtain (3.5).

Suppose now that  $H$  is Lipschitz-like with respect to  $p$  uniformly in  $x$  around  $(\bar{x}, \bar{p}, 0)$ . Then there exist  $l, \alpha, \beta, \tau > 0$  such that, for every  $x \in B(\bar{x}, \alpha)$  and every  $p_1, p_2 \in B(\bar{p}, \beta)$ ,

$$H(x, p_1) \cap B(0, \tau) \subset H(x, p_2) + l \|p_1 - p_2\| \mathbb{D}_Y. \quad (3.11)$$

Take  $\bar{\alpha} := \min\{\alpha, \bar{r}\}$ ,  $\bar{\beta} = \min\{\beta, \bar{t}\}$ ,  $p_1, p_2 \in B(\bar{p}, \bar{\beta})$  and  $x \in S(p_1) \cap B(\bar{x}, \bar{\alpha})$ . Then, using (3.11) and (3.5),

$$d(x, S(p_2)) \leq c^{-1}d(0, H(x, p_2)) \leq c^{-1}l \|p_1 - p_2\|.$$

Hence, because  $l$  can be chosen arbitrarily close to  $\widehat{\text{lip}}_p H((\bar{x}, \bar{p}), 0)$ , it follows that  $S$  is Lipschitz-like around  $(\bar{p}, \bar{x})$  and  $\text{lip } S(\bar{p}, \bar{x}) \leq c^{-1} \widehat{\text{lip}}_p H((\bar{x}, \bar{p}), 0)$ . The proof is now complete.  $\square$

Remark that, for the second parts of the above items, we can replace the inner semicontinuity assumption of  $H$  at  $(\bar{x}, \bar{p}, 0)$  with the condition of inner semicontinuity with respect to the variable in which  $H$  is not Lipschitz. See, also, the proof of Proposition 4.1. It is worth to be mentioned that one can obtain in the second parts even a kind of graphical regularity, following the technique from [6, Theorem 5.2] (see, also, [7]).

## 4 Applications

This section is dedicated to the investigation of the case where the mapping  $H$  is given as a sum of two set-valued maps in the sense we shall precise. However, we start with an application of Theorem 3.6 and we get an implicit multifunction result which generalizes [2, Theorem 3.5] to the case where the set-valued map  $\Gamma$  is constructed using the sum between a function and a multifunction. Also, in the virtue of Corollary 3.5, we can conclude that this result generalizes also [2, Lemma 3.1].

**Proposition 4.1** *Let  $X, Y, Z, W$  be Banach spaces,  $F : X \times Y \rightrightarrows Z$  be a multifunction,  $g : W \rightarrow Z$  be a function and  $(\bar{x}, \bar{y}, \bar{z}, \bar{w}) \in X \times Y \times Z \times W$  be such that  $\bar{z} := -g(\bar{w}) \in F(\bar{x}, \bar{y})$ . Consider next the implicit multifunction  $\Gamma : Y \times W \rightrightarrows X$  defined by*

$$\Gamma(y, w) := \{x \in X \mid 0 \in F(x, y) + g(w)\}.$$

*Suppose that the following conditions are satisfied:*

- (i)  *$F$  is Lipschitz-like with respect to  $y$  uniformly in  $x$  around  $((\bar{x}, \bar{y}), \bar{z})$  with constant  $\eta \geq 0$ ;*
- (ii)  *$F$  is metrically regular with respect to  $x$  uniformly in  $y$  around  $((\bar{x}, \bar{y}), \bar{z})$  with constant  $k > 0$ ;*
- (iii)  *$F(\cdot, y)$  is inner semicontinuous at  $(\bar{x}, \bar{z})$  for every  $y$  in a neighborhood of  $\bar{y}$ ;*
- (iv)  *$g$  is locally Lipschitzian around  $\bar{w}$  with constant  $\lambda$ .*

*Then there exists  $\alpha > 0$  such that for every  $(y, w), (y', w') \in D(\bar{y}, \alpha) \times D(\bar{w}, \alpha)$  and for every  $\varepsilon > 0$ ,*

$$\Gamma(y', w') \cap D(\bar{x}, \alpha) \subset \Gamma(y, w) + (k + \varepsilon)(\eta \|y - y'\| + \lambda \|w - w'\|)\mathbb{D}_X.$$

*In particular,  $\Gamma$  is Lipschitz-like around  $((\bar{y}, \bar{w}), \bar{x})$  with the following estimate*

$$\text{lip } \Gamma((\bar{y}, \bar{w}), \bar{x}) \leq \widehat{\text{reg}}_x F((\bar{x}, \bar{y}), \bar{z}) \cdot \max\{\widehat{\text{lip}}_y F(\bar{x}, \bar{y}, \bar{z}), \text{lip } g(\bar{w})\}.$$

**Proof.** Define  $P := Y \times W$  and  $H : X \times P \rightrightarrows Z$  by  $H(x, (y, w)) := F(x, y) + g(w)$ .

We want to prove first that  $F$  is inner semicontinuous at  $(\bar{x}, \bar{y}, \bar{z})$ . For this, take arbitrarily  $\delta > 0$ ; we know that  $F(\bar{x}, \bar{y}) \cap B(\bar{z}, \delta) \neq \emptyset$ . Using (i), we find  $a > 0$  such that, for every  $y \in B(\bar{y}, a)$ ,

$$\bar{z} \in F(\bar{x}, \bar{y}) \subset F(\bar{x}, y) + \eta \|y - \bar{y}\| \mathbb{D}_Z.$$

Then, for every  $y$  such that  $\|y - \bar{y}\| < \min\{a, \eta^{-1}\delta\}$ ,  $F(\bar{x}, y) \cap B(\bar{z}, \delta) \neq \emptyset$ . Using now (i), we get that for  $y$  sufficiently close to  $\bar{y}$ , there exists a neighborhood  $U$  of  $\bar{x}$  such that, for every  $x \in U$ ,  $F(x, y) \cap B(\bar{z}, \delta) \neq \emptyset$ . In other words,  $F$  is inner semicontinuous at  $(\bar{x}, \bar{y}, \bar{z})$ . By this property and the continuity of  $g$ , one can easily prove that  $H$  is inner semicontinuous at  $(\bar{x}, (\bar{y}, \bar{z}), 0)$ . Moreover,

$$\Gamma(y, w) = \{x \in X \mid 0 \in H(x, (y, w))\}$$

and denote  $H_{(y, w)}(\cdot) := H(\cdot, (y, w))$ . Because for every  $y$  close to  $\bar{y}$  we know from (ii) and Proposition 2.2 that  $F_y$  is  $k^{-1}$ -open at points from its graph around  $(\bar{x}, \bar{z})$  and because  $g$  does not depend of  $x$ , we can conclude that there exist  $\varepsilon, \gamma > 0$  such that for every  $\rho \in ]0, \varepsilon[$ , every  $(y, w) \in B(\bar{y}, \gamma) \times B(\bar{w}, \gamma)$  and every  $(x, z) \in \text{Gr } H_{(y, w)} \cap [B(\bar{x}, \gamma) \times B(0, \gamma)]$ ,

$$B(z, k^{-1}\rho) \subset H_{(y, w)}(B(0, \rho)).$$

But this shows, applying Theorem 3.6, that there exist  $\beta > 0$  such that, for every  $(x, y, w) \in B(\bar{x}, \beta) \times B(\bar{y}, \beta) \times B(\bar{w}, \beta)$ ,

$$d(x, \Gamma(y, w)) \leq kd(0, H(x, (y, w))). \quad (4.1)$$

We want to prove that there exists  $c > 0$  such that, for every  $x \in B(\bar{x}, c)$ ,  $(y, w), (y', w') \in B(\bar{y}, c) \times B(\bar{w}, c)$ ,

$$H(x, (y, w)) \cap D(0, c) \subset H(x, (y', w')) + (\eta \|y - y'\| + \lambda \|w - w'\|) \mathbb{D}_Z. \quad (4.2)$$

In particular, we will prove that  $H$  is Lipschitz-like with respect to  $(y, w)$  uniformly in  $x$  around  $(\bar{x}, (\bar{y}, \bar{z}), 0)$ .

Because of (i), we know that there exists  $a > 0$  such that for every  $x \in B(\bar{x}, a)$  and every  $y, y' \in B(\bar{y}, a)$ ,

$$F(x, y) \cap D(\bar{z}, a) \subset F(x, y') + \eta \|y - y'\| \mathbb{D}_Z. \quad (4.3)$$

Also, because of (iv), we can find  $b > 0$  such that for every  $w, w' \in B(\bar{w}, b)$ ,

$$g(w) \in g(w') + \lambda \|w - w'\| \mathbb{D}_Z. \quad (4.4)$$

Choose now  $c > 0$  such that  $c < \min\{(\lambda+1)^{-1}a, b\}$  and take arbitrarily  $x \in B(\bar{x}, c)$ ,  $(y, w), (y', w') \in B(\bar{y}, c) \times B(\bar{w}, c)$ . Furthermore, choose  $z \in H(x, (y, w)) \cap D(0, c)$ . Then  $z - g(w) \in F(x, y)$  and because of (4.4), we know that  $-g(w) \in D(\bar{z}, \lambda \|w - \bar{w}\|) \subset B(\bar{z}, \lambda c)$ , whence  $z - g(w) \in B(\bar{z}, (\lambda+1)c) \subset B(\bar{z}, a)$ . One can use now (4.3) to obtain that  $z - g(w) \in F(x, y') + \eta \|y - y'\| \mathbb{D}_Z$ . Adding with (4.4), one finally gets (4.2).

Take now  $\alpha < \min\{\beta, c\}$ ,  $(y, w), (y', w') \in B(\bar{y}, \alpha) \times B(\bar{w}, \alpha)$ ,  $x \in \Gamma(y, w) \cap D(\bar{x}, \alpha)$  and arbitrary  $\varepsilon > 0$ . Then, using (4.1) and (4.2),

$$\begin{aligned} d(x, \Gamma(y', w')) &\leq kd(0, H(x, (y', w'))) \leq k(\eta \|y - y'\| + \lambda \|w - w'\|) \\ &< (k + \varepsilon)(\eta \|y - y'\| + \lambda \|w - w'\|), \end{aligned}$$

which completes the proof.  $\square$

Note that all the conclusion of Theorem 3.6 (i), but the estimation (3.5), could be obtained as a consequence of Proposition 4.1, taking  $g \equiv 0$ . We note as well that for  $W := Z$  and  $g(z) := -z$  for every  $z$  we can get an even more general implicit multifunction result (see [6, Theorem 5.2]).

The next technical notion will be used in the sequel, mainly to prove a Lipschitz-like property of the sum between two multifunctions. In this way (in contrast to [10, Corollary 18]), we avoid the strong requirements of the single-valuedness and full Lipschitz property of the field map  $G$  at the reference point.

**Definition 4.2** *Let  $F : X \rightrightarrows Y$ ,  $G : X \rightrightarrows Y$  be two multifunctions and  $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Y$  such that  $\bar{y} \in F(\bar{x})$ ,  $\bar{z} \in G(\bar{x})$ . We say that the multifunction  $(F, G)$  is locally sum-stable around  $(\bar{x}, \bar{y}, \bar{z})$  if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that, for every  $x \in B(\bar{x}, \delta)$  and every  $w \in (F + G)(x) \cap B(\bar{y} + \bar{z}, \delta)$ , there exist  $y \in F(x) \cap B(\bar{y}, \varepsilon)$  and  $z \in G(x) \cap B(\bar{z}, \varepsilon)$  such that  $w = y + z$ .*

This definition is illustrated, at a first glance, by two simple examples. First example displays a simple situation where this condition holds true. Note that  $\mathbb{R}$  and  $\mathbb{Q}$  denote the fields of reals and rationals, respectively.

**Example 4.3** Let  $F : \mathbb{R} \rightrightarrows \mathbb{R}$  given, for any  $x \in \mathbb{R}$ , by

$$F(x) := [0, |x|].$$

Take now  $G := F$ . It is easy to see that  $(F, G)$  is locally sum-stable around any  $(x, y, z) \in \text{Gr}(F, G)$ .

The next example describes a situation where the sum-stable condition does not hold.

**Example 4.4** Let  $F : \mathbb{R} \rightrightarrows \mathbb{R}$  be given by

$$F(x) := \begin{cases} [-1, 1], & \text{if } x \in \mathbb{R} \setminus \{1\} \\ 2, & \text{if } x = 1 \end{cases}$$

and  $G : \mathbb{R} \rightrightarrows \mathbb{R}$  given, for any  $x \in \mathbb{R}$ , by  $G(x) := -F(x)$ . Then one can easily see that  $(F, G)$  is not sum-stable at  $(\bar{x}, \bar{y}, \bar{z}) := (1, 2, -2)$  because  $0 \in (F + G)(x)$  for every  $x \in \mathbb{R}$  but if  $x \neq 1$ , then we cannot write 0 as a sum between an element in  $F(x) \cap B(2, 2^{-1})$  and an element in  $G(x) \cap B(-2, 2^{-1})$ .

Remind that a multifunction  $F : X \rightrightarrows Y$  is said to be Lipschitz around  $(\bar{x}, \bar{y}) \in \text{Gr } F$  with constant  $L > 0$  if there exists a neighborhood  $U \in \mathcal{V}(\bar{x})$  such that, for every  $x, u \in U$ ,

$$F(x) \subset F(u) + L \|x - u\| \mathbb{D}_Y. \quad (4.5)$$

Of course, this property is (much) stronger than the Lipschitz-like property, having the great advantage to be stable at summation. More precisely, if  $F, G$  are two multifunctions which are Lipschitz around some points  $(\bar{x}, \bar{y}) \in \text{Gr } F, (\bar{x}, \bar{z}) \in \text{Gr } G$ , then  $F + G$  is Lipschitz around  $(\bar{x}, \bar{y} + \bar{z})$ .

The next, more elaborated example, shows that the Lipschitz property of both multifunctions does not ensure the sum-stable property.

**Example 4.5** Let  $F : \mathbb{R} \rightrightarrows \mathbb{R}$  be given by  $F(x) := [1, +\infty[ \cap \mathbb{Q}$  for every  $x \in \mathbb{R}$  and  $G : \mathbb{R} \rightrightarrows \mathbb{R}$  given by

$$G(x) := \begin{cases} ]-\infty, -1], & \text{if } x \in \mathbb{R} \setminus \{1 - \frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\}\} \\ ]-\infty, -1] \cap \mathbb{Q} \cup \{-2 + \frac{\sqrt{2}}{n}\}, & \text{if } x = 1 - \frac{1}{n}, n \in \mathbb{N} \setminus \{0\}. \end{cases}$$

It is easy to verify that both  $F$  and  $G$  are Lipschitz around  $(\bar{x}, \bar{y}) := (1, 1)$  and  $(\bar{x}, \bar{z}) := (1, -1)$ , respectively, because, for example, if we add to the set  $]-\infty, -1] \cap \mathbb{Q} \cup \{-2 + \frac{\sqrt{2}}{n}\}$  any ball, then we cover all the interval  $]-\infty, -1]$ . Fix  $\varepsilon := 2^{-1}$  and take arbitrarily  $\delta > 0$ . Then choose  $x_n := 1 - \frac{1}{n}$ ,  $n \in \mathbb{N} \setminus \{0\}$  and observe that  $x_n \in ]1 - \delta, 1 + \delta[$  and

$$w_n := \frac{\sqrt{2}}{n} \in (F + G)(x_n) \cap B(0, \delta)$$

for any  $n$  sufficiently large. But  $w_n$  can be obtained only by the sum between  $2 \in F(x_n)$  and  $-2 + \frac{\sqrt{2}}{n} \in G(x_n)$  and both these values are not in the balls  $B(\bar{y}, 2^{-1})$  and  $B(\bar{z}, 2^{-1})$  respectively. Therefore,  $(F, G)$  is not sum-stable at  $(\bar{x}, \bar{y}, \bar{z}) = (1, 1, -1)$ .

Next proposition indicates a first general situation where the local-sum stability holds. Recall that a function  $f : X \rightarrow Y$  is calm at  $\bar{x}$  if there exist  $\alpha, l > 0$  such that, for every  $x \in B(\bar{x}, \alpha)$ ,

$$\|f(x) - f(\bar{x})\| \leq l \|x - \bar{x}\|.$$

**Proposition 4.6** *Let  $f : X \rightarrow Y$  be a function,  $G : X \rightrightarrows Y$  be a multifunction and  $(\bar{x}, \bar{p}) \in X \times P$  such that  $0 \in f(\bar{x}) + G(\bar{x})$ . If  $f$  is calm at  $\bar{x}$ , then  $(f, G)$  is locally sum-stable around  $(\bar{x}, f(\bar{x}), -f(\bar{x}))$ .*

**Proof.** Suppose that the constants from the calmness property of  $f$  are the same as above, take arbitrarily  $\varepsilon > 0$  and choose  $\delta \in ]0, \min\{2^{-1}\varepsilon, (2l)^{-1}\varepsilon, \alpha\}[$ . Pick now  $x \in B(\bar{x}, \delta)$  and  $w \in (f + G)(x) \cap B(0, \delta)$ . Then  $w - f(x) \in G(x)$ . Moreover,  $\|f(x) - f(\bar{x})\| \leq l\|x - \bar{x}\| < l\delta < 2^{-1}\varepsilon$ , so  $f(x) \in B(f(\bar{x}), \varepsilon)$ . Also,  $w - f(x) \in B(0, \delta) + B(-f(\bar{x}), 2^{-1}\varepsilon) \subset B(-f(\bar{x}), \varepsilon)$ . The proof is now complete.  $\square$

The next lemma is the main motivation for introducing the sum-stable property.

**Lemma 4.7** *Let  $F : X \rightrightarrows Y$ ,  $G : X \rightrightarrows Y$  be two multifunctions. Suppose that  $F$  is Lipschitz-like around  $(\bar{x}, \bar{y}) \in \text{Gr } F$ , that  $G$  is Lipschitz-like around  $(\bar{x}, \bar{z}) \in \text{Gr } G$  and that  $(F, G)$  is locally sum-stable around  $(\bar{x}, \bar{y}, \bar{z})$ . Then the multifunction  $F + G$  is Lipschitz-like around  $(\bar{x}, \bar{y} + \bar{z})$ . Moreover, the following relation holds true*

$$\text{lip}(F + G)(\bar{x}, \bar{y} + \bar{z}) \leq \text{lip } F(\bar{x}, \bar{y}) + \text{lip } G(\bar{x}, \bar{z}). \quad (4.6)$$

**Proof.** Using the Lipschitz-like properties of  $F$  and  $G$ , one can find  $\alpha, l, k > 0$  such that, for every  $x, u \in B(\bar{x}, \alpha)$ ,

$$F(x) \cap B(\bar{y}, \alpha) \subset F(u) + l\|x - u\| \mathbb{D}_Y, \quad (4.7)$$

$$G(x) \cap B(\bar{z}, \alpha) \subset G(u) + k\|x - u\| \mathbb{D}_Y. \quad (4.8)$$

But using the local sum-stability for  $\varepsilon := \alpha > 0$ , we can find  $\delta \in ]0, \alpha[$  such that, for every  $x \in B(\bar{x}, \delta)$  and every  $w \in (F + G)(x) \cap B(\bar{y} + \bar{z}, \delta)$ , there exist  $y \in F(x) \cap B(\bar{y}, \alpha)$  and  $z \in G(x) \cap B(\bar{z}, \alpha)$  such that  $w = y + z$ . Consequently, using (4.7) and (4.8), for every  $u \in B(\bar{x}, \delta)$ ,  $w \in (F + G)(u) + (l + k)\|x - u\| \mathbb{D}_Y$ . The relation (4.6) follows from the fact that constants  $l$  and  $k$  can be chosen arbitrarily close to  $\text{lip } F(\bar{x}, \bar{y})$  and  $\text{lip } G(\bar{x}, \bar{z})$ , respectively.  $\square$

We would like to continue with our examples above in order to illustrate the fact that sum-stable property is essential in Lemma 4.7. Basically, we need an example of two Lipschitz-like multifunctions for which the sum is not Lipschitz-like at the reference point.

**Example 4.8** Let  $F : \mathbb{R} \rightrightarrows \mathbb{R}$  be given by

$$F(x) := \begin{cases} [1, 2], & \text{if } x \in \mathbb{R} \setminus \{1\} \\ [1, 2] \cup \{0\}, & \text{if } x = 1 \end{cases}$$

and  $G : \mathbb{R} \rightrightarrows \mathbb{R}$  given by  $G(x) := [1, 2]$  for every  $x \in \mathbb{R}$ .

As above, it is easy to see that both  $F$  and  $G$  are Lipschitz-like around  $(\bar{x}, \bar{y}) = (1, 1)$  and  $(\bar{x}, \bar{z}) = (1, 1)$ , respectively. But the multifunction  $F + G : \mathbb{R} \rightrightarrows \mathbb{R}$ , given by

$$(F + G)(x) = \begin{cases} [2, 4], & \text{if } x \in \mathbb{R} \setminus \{1\} \\ [1, 4], & \text{if } x = 1 \end{cases},$$

is not Lipschitz-like around  $(\bar{x}, \bar{y} + \bar{z}) = (1, 2)$ . Indeed, suppose by contradiction that there exists  $L > 0$  and  $\alpha \in ]0, \min\{1, L^{-1}\}[$  such that for any  $x, u \in [1 - \alpha, 1 + \alpha]$

$$(F + G)(x) \cap [2 - \alpha, 2 + \alpha] \subset (F + G)(u) + L\|x - u\|[-1, 1]. \quad (4.9)$$

Consider now  $x := 1$  and  $u := 1 - \alpha^2$  such that  $x, u \in [1 - \alpha, 1 + \alpha]$ . Clearly,

$$2 - \alpha \in (F + G)(x) \cap [2 - \alpha, 2 + \alpha].$$



Following (4.9), we should have:

$$2 - \alpha \in (F + G)(1 - \alpha^2) + L\alpha^2[-1, 1]$$

and, in particular,

$$2 - \alpha \in [2 - L\alpha^2, 4 + L\alpha^2].$$

But this requires that

$$\alpha \leq L\alpha^2,$$

which contradicts the choice of  $\alpha$ . The contradiction shows that we cannot have the Lipschitz-like property of the sum.

Observe that, in the virtue of Lemma 4.7,  $(F, G)$  cannot be locally-sum stable around  $(1, 1, 1)$ . Indeed, take  $\varepsilon \in ]0, 2^{-1}[$ . Then for every  $\delta > 0$ , choose  $n \in \mathbb{N}$  such that  $n > \max\{\delta, 1\}$ . Taking now  $x_\delta := 1 \in ]1 - \delta, 1 + \delta[$  and  $w_\delta := 2 - n^{-1}\delta \in (F + G)(x_\delta) \cap ]2 - \delta, 2 + \delta[$ , one can easily see that, for every  $y \in F(x_\delta) \cap ]1 - \varepsilon, 1 + \varepsilon[$  and every  $z \in G(x_\delta) \cap ]1 - \varepsilon, 1 + \varepsilon[$ ,  $w_\delta < y + z$ .

Next, we adapt the definition of local-sum stability to the parametric case, in order to use this notion in the general context of variational systems.

**Definition 4.9** *Let  $F : X \times P \rightrightarrows Y$ ,  $G : X \rightrightarrows Y$  be two multifunctions and  $(\bar{x}, \bar{p}, \bar{y}, \bar{z}) \in X \times P \times Y \times Y$  such that  $\bar{y} \in F(\bar{x}, \bar{p})$ ,  $\bar{z} \in G(\bar{x})$ . We say that the multifunction  $(F, G)$  is locally sum-stable around  $(\bar{x}, \bar{p}, \bar{y}, \bar{z})$  if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that, for every  $(x, p) \in B(\bar{x}, \delta) \times B(\bar{p}, \delta)$  and every  $w \in (F_p + G)(x) \cap B(\bar{y} + \bar{z}, \delta)$ , there exist  $y \in F_p(x) \cap B(\bar{y}, \varepsilon)$  and  $z \in G(x) \cap B(\bar{z}, \varepsilon)$  such that  $w = y + z$ .*

Similarly to Proposition 4.6, one can easily prove the next (adapted) result.

**Proposition 4.10** *Let  $f : X \times P \rightarrow Y$  be a function,  $G : X \rightrightarrows Y$  be a multifunction and  $(\bar{x}, \bar{p}) \in X \times P$  such that  $0 \in f(\bar{x}, \bar{p}) + G(\bar{x})$ . If  $f$  is calm at  $(\bar{x}, \bar{p})$ , then  $(f, G)$  is locally sum-stable around  $(\bar{x}, \bar{p}, f(\bar{x}, \bar{p}), -f(\bar{x}, \bar{p}))$ .*

Also, Lemma 4.7 has the following variant in the parametric case.

**Lemma 4.11** *Let  $F : X \times P \rightrightarrows Y$ ,  $G : X \rightrightarrows Y$  be two multifunctions. Suppose that  $F$  is Lipschitz-like with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), \bar{y}) \in \text{Gr } F$ , that  $G$  is Lipschitz-like around  $(\bar{x}, \bar{z}) \in \text{Gr } G$  and that  $(F, G)$  is locally sum-stable around  $(\bar{x}, \bar{p}, \bar{y}, \bar{z})$ . Then the multifunction  $H : X \times P \rightrightarrows Y$  given by  $H(x, p) := F(x, p) + G(x)$  is Lipschitz-like with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), \bar{y} + \bar{z})$ . Moreover, the following relation holds true*

$$\widehat{\text{lip}}_x H((\bar{x}, \bar{p}), \bar{y} + \bar{z}) \leq \widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y}) + \text{lip } G(\bar{x}, \bar{z}). \quad (4.10)$$

The following result deduces the metric regularity of  $S$  under appropriate assumptions on the multifunctions  $F$  and  $G$ , which arrive naturally from Theorem 3.6. Namely, part of these assumptions are stated in order to ensure the Lipschitz-like property of the sum multifunction  $H$  with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), 0)$ , with (4.10) satisfied. Using (3.8), we expect to have that  $\text{reg } S(\bar{p}, \bar{x}) \leq c^{-1} \widehat{\text{lip}}_x H((\bar{x}, \bar{p}), 0)$ , where  $c > 0$  is the rate of linear openness with respect to  $p$  of  $H$ , but is easy to see that  $c$  must be  $\text{lop}_p F((\bar{x}, \bar{p}), \bar{y}) = (\widehat{\text{reg}}_p F((\bar{x}, \bar{p}), \bar{y}))^{-1}$ . Therefore, (4.11) below holds in a natural way.

**Theorem 4.12** *Let  $X, Y, P$  be Banach spaces,  $F : X \times P \rightrightarrows Y$ ,  $G : X \rightrightarrows Y$  be two set-valued maps and  $(\bar{x}, \bar{p}, \bar{y}) \in X \times P \times Y$  such that  $\bar{y} \in F(\bar{x}, \bar{p})$  and  $-\bar{y} \in G(\bar{x})$ . Suppose that the following assumptions are satisfied:*

- (i)  $(F, G)$  is locally sum-stable around  $(\bar{x}, \bar{p}, \bar{y}, -\bar{y})$ ;
- (ii)  $F(\cdot, x)$  is inner semicontinuous around  $(\bar{p}, \bar{y})$  for every  $x$  in a neighborhood of  $\bar{x}$ ;
- (iii)  $F$  is Lipschitz-like with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), \bar{y})$ ;
- (iv)  $F$  is metrically regular with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), \bar{y})$ ;
- (v)  $G$  is Lipschitz-like around  $(\bar{x}, -\bar{y})$ .

*Then  $S$  is metrically regular around  $(\bar{p}, \bar{x})$ . Moreover, the next relation holds*

$$\text{reg } S(\bar{p}, \bar{x}) \leq \widehat{\text{reg}}_p F((\bar{x}, \bar{p}), \bar{y}) \cdot [\widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y}) + \text{lip } G(\bar{x}, -\bar{y})]. \quad (4.11)$$

**Proof.** Define  $H : X \times P \rightrightarrows Y$  by

$$H(x, p) := F(x, p) + G(x). \quad (4.12)$$

Using Lemma 4.11, we know that  $H$  is Lipschitz-like with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), 0)$  and the relation (4.10) holds for  $\bar{z} := -\bar{y}$ .

Also, similarly to the proof of Proposition 4.1, we can show that  $F$  is inner semicontinuous at  $((\bar{x}, \bar{p}), \bar{y})$ . Let us take  $\varepsilon > 0$ . Using (v), there exists  $l_G > 0$  and  $\alpha \in ]0, (2l_G)^{-1}\varepsilon[$  s.t. for every  $x \in B(\bar{x}, \alpha)$

$$-\bar{y} \in G(x) + l_G \|x - \bar{x}\| \mathbb{D}_Y.$$

Therefore, we can find  $-z_x \in G(x)$  s.t.

$$\|z - \bar{y}\| \leq l_G \|x - \bar{x}\| < 2^{-1}\varepsilon.$$

By the inner-semicontinuity of  $F$  we can find  $\mu \in ]0, 2^{-1}\varepsilon[$  s.t. for every  $(x, p) \in B(\bar{x}, \mu) \times B(\bar{p}, \mu)$  there exists

$$y_{(x,p)} \in F(x, p) \cap B(\bar{y}, 2^{-1}\varepsilon).$$

Then, we get that for every  $(x, p) \in B(\bar{x}, \min\{\alpha, \mu\}) \times B(\bar{p}, \mu)$  we can find, as above,  $z_x, y_{(x,p)}$  s.t.

$$y_{(x,p)} - z_x \in [F(x, p) + G(x)] \cap B(0, \varepsilon)$$

since

$$\|z_x - y_{(x,p)}\| \leq \|z_x - \bar{y}\| + \|\bar{y} - y_{(x,p)}\| < 2^{-1}\varepsilon + 2^{-1}\varepsilon = \varepsilon.$$

This shows that  $H$  is inner semicontinuous at  $((\bar{x}, \bar{p}), 0)$ .

Using now Proposition 2.2, (iv) is equivalent to the fact that  $F$  is open at linear rate with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), \bar{y})$  and  $\widehat{\text{lop}}_p F((\bar{x}, \bar{p}), \bar{y}) = (\widehat{\text{reg}}_p F((\bar{x}, \bar{p}), \bar{y}))^{-1}$ . Adding the fact that  $G$  does not depend of  $p$ , we obtain that  $H$  is open at linear rate with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), 0)$  and  $\widehat{\text{lop}}_p H((\bar{x}, \bar{p}), \bar{y}) = (\widehat{\text{reg}}_p F((\bar{x}, \bar{p}), \bar{y}))^{-1}$ .

Now the result easily follows from the second item of Theorem 3.6.  $\square$

Theorem 4.12 is a generalization of [1, Theorem 3.3 (i)], concerning the direct implication. On the other hand, the next result uses the ideas of the proof of Theorem 3.6 and, essentially, the estimations from Theorem 3.3. For several technical reasons it is not possible to give a direct and easy proof based on the main results, but, nevertheless, the proof uses the very same ideas and arguments arranged in the specific context of this result.

Let us emphasize that, once again, the Lipschitz modulus of  $S$  has a form which can be easily developed from the previous facts. Namely, one can expect that  $\text{lip } S(\bar{p}, \bar{x}) \leq c^{-1} \widehat{\text{lip}}_p H((\bar{x}, \bar{p}), 0) = c^{-1} \widehat{\text{lip}}_p F((\bar{x}, \bar{p}), \bar{y})$ , where  $c > 0$  must be the rate of linear openness of  $H$  with respect to  $x$ . But because for every  $p$  in a neighborhood of  $\bar{p}$ , and for appropriate  $x, y$ ,  $\widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y})$  seems to be very close to  $\text{lip}(F_p)(x, y) = \text{lip}(-F_p)(x, -y) = (\text{lop}(-F_p)^{-1}(-y, x))^{-1}$ , because  $\text{reg } G(\bar{x}, -\bar{y}) = (\text{lop } G(\bar{x}, -\bar{y}))^{-1}$  and, essentially, because  $\widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y}) \cdot \text{reg } G(\bar{x}, -\bar{y}) < 1$ , then one can expect that  $G - (-F_p)$  might be open at the linear rate  $c = \text{lop } G(\bar{x}, -\bar{y}) - \widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y}) = (\text{reg } G(\bar{x}, -\bar{y}))^{-1} - \widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y})$  at points close to  $((\bar{x}, \bar{p}), 0)$ . Whence, once again, one can have an intuitive approach in getting (4.13) below.

**Theorem 4.13** *Let  $X, Y, P$  be Banach spaces,  $F : X \times P \rightrightarrows Y$ ,  $G : X \rightrightarrows Y$  be two set-valued maps and  $(\bar{x}, \bar{p}, \bar{y}) \in X \times P \times Y$  such that  $\bar{y} \in F(\bar{x}, \bar{p})$  and  $-\bar{y} \in G(\bar{x})$ . Suppose that the following assumptions are satisfied:*

- (i)  $(F, G)$  is locally sum-stable with respect to  $x$  uniformly in  $p$  around  $(\bar{x}, \bar{p}, \bar{y}, -\bar{y})$ ;
- (ii) for every  $p$  in a neighborhood of  $\bar{p}$ ,  $\text{Gr } F_p$  is closed;
- (iii)  $\text{Gr } G$  is closed;
- (iv)  $F$  is Lipschitz-like around  $((\bar{x}, \bar{p}), \bar{y})$ ;
- (v)  $G$  is metrically regular around  $(\bar{x}, -\bar{y})$ ;
- (vi)  $G$  is inner semicontinuous at  $(\bar{x}, -\bar{y})$ ;
- (vii)  $\widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y}) \cdot \text{reg } G(\bar{x}, -\bar{y}) < 1$ .

*Then  $S$  is Lipschitz-like around  $(\bar{p}, \bar{x})$ . Moreover, the next relation is satisfied*

$$\text{lip } S(\bar{p}, \bar{x}) \leq \frac{\text{reg } G(\bar{x}, -\bar{y}) \cdot \widehat{\text{lip}}_p F((\bar{x}, \bar{p}), \bar{y})}{1 - \widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y}) \cdot \text{reg } G(\bar{x}, -\bar{y})}. \quad (4.13)$$

**Proof.** Take  $m > \text{reg } G(\bar{x}, -\bar{y})$  and  $l > \widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y})$  such that  $m \cdot l < 1$ .

Observe first that in our assumptions  $F(\cdot, \cdot) + G(\cdot)$  is inner semicontinuous at  $((\bar{x}, \bar{p}), 0)$ . Indeed, (iv) ensures that  $F$  is inner semicontinuous at  $((\bar{x}, \bar{p}), \bar{y})$ : consider  $\varepsilon > 0$  and take a positive  $\delta$  strictly smaller than the minimum between  $(2L)^{-1}\varepsilon$  (where  $L$  is the Lipschitz constant of  $F$ ) and the radii of the balls of  $\bar{x}$  and  $\bar{p}$  involved in the Lipschitz condition. Then for every  $(x, p) \in B(\bar{x}, \delta) \times B(\bar{p}, \delta)$  there exists  $y \in F(x, p)$  with

$$\|y - \bar{y}\| \leq L(\|x - \bar{x}\| + \|p - \bar{p}\|) < \varepsilon.$$

This proves that  $F$  is inner semicontinuous at  $((\bar{x}, \bar{p}), \bar{y})$ . This fact in addition with (vi) ensures that  $F(\cdot, \cdot) + G(\cdot)$  is inner semicontinuous at  $((\bar{x}, \bar{p}), 0)$ . Indeed, take  $\varepsilon > 0$ . There exists  $\delta_1 > 0$  s.t. for every  $(x, p) \in B(\bar{x}, \delta_1) \times B(\bar{p}, \delta_1)$ ,  $F(x, p) \cap B(\bar{y}, 2^{-1}\varepsilon) \neq \emptyset$  and there exists  $\delta_2 > 0$  s.t. for every  $x \in B(\bar{x}, \delta_2)$ ,  $G(x) \cap B(-\bar{y}, 2^{-1}\varepsilon) \neq \emptyset$ . Take  $\delta = \min\{\delta_1, \delta_2\}$ . Now, for  $(x, p) \in B(\bar{x}, \delta) \times B(\bar{p}, \delta)$ , we can find  $y \in F(x, p) \cap B(\bar{y}, 2^{-1}\varepsilon)$  and  $z \in G(x) \cap B(-\bar{y}, 2^{-1}\varepsilon)$ , whence

$$y + z \in [F(x, p) + G(x)] \cap B(0, \varepsilon)$$

and the claim is proved.

Now, we intend to prove that there exist  $\tau, t > 0$  such that, for every  $(x, p) \in B(\bar{x}, \tau) \times B(\bar{p}, t)$ ,

$$d(x, S(p)) \leq m(1 - lm)^{-1}d(0, F(x, p) + G(x)). \quad (4.14)$$

Using the assumption (iv), one can find  $r_1, t_1 > 0$  such that, for every  $p \in B(\bar{p}, t_1)$ , and every  $x, x' \in B(\bar{x}, r_1)$ ,

$$F_p(x) \cap B(\bar{y}, s_1) \subset F_p(x') + l \|x - x'\| \mathbb{D}_Y. \quad (4.15)$$

But this shows, as one can see next, that for every  $p \in B(\bar{p}, t_1)$ ,  $F_p$  is  $l$ -pseudocalm at every  $(x, y) \in \text{Gr } F_p \cap [B(\bar{x}, 2^{-1}r_1) \times B(\bar{y}, s_1)]$ . Indeed, take  $x \in B(\bar{x}, 2^{-1}r_1)$  and  $y \in F_p(x) \cap B(\bar{y}, s_1)$ . Then for every  $x' \in B(x, 2^{-1}r_1)$ , we have from (4.15) that there exists  $y' \in F_p(x')$  such that

$$d(y, F(x')) \leq \|y - y'\| \leq l \|x - x'\|,$$

which proves the desired assertion. Hence, we conclude in view of Proposition 2.4 that for every  $p \in B(\bar{p}, t_1)$ ,  $F_p^{-1}$  is  $l^{-1}$ -open at every  $(y, x) \in \text{Gr } F_p^{-1} \cap [B(\bar{y}, s_1) \times B(\bar{x}, 2^{-1}r_1)]$ .

Also, from (v), we know that there exist  $r_2, s_2 > 0$  such that, for every  $(u, v) \in \text{Gr } G \cap [B(\bar{x}, r_2) \times B(\bar{y}, s_2)]$ ,  $G$  is metrically hemiregular at  $(u, v)$  with constant  $m$ , whence is open at linear rate  $m^{-1}$  at  $(u, v)$ .

Use now the property from (i) for  $\min\{2^{-1}s_1, 2^{-1}s_2\}$  instead of  $\varepsilon$  and find  $\delta$  such that the assertion from Definition 4.9 is true.

Take now  $\rho \in ]0, \min\{(m^{-1} - l)^{-1}\delta, 4^{-1}r_1, 2^{-1}r_2, 2^{-1}l^{-1}s_1, 2^{-1}ms_2\}[$ . Using the inner semicontinuity of  $F + G$ , one can find  $\nu, t_2 \in ]0, \delta[$  such that, for every  $(x, p) \in B(\bar{x}, \nu) \times B(\bar{p}, t_2)$ ,

$$[F(x, p) + G(x)] \cap B(0, (m^{-1} - l)\rho) \neq \emptyset. \quad (4.16)$$

Suppose that  $B(\bar{p}, t_3)$  is the neighborhood which appears in (ii), denote by  $t := \min\{t_1, t_2, t_3\}$ ,  $\tau := \min\{\nu, 4^{-1}r_1, 2^{-1}r_2\}$ , and take  $(x, p) \in B(\bar{x}, \tau) \times B(\bar{p}, t)$ .

If  $0 \in F(x, p) + G(x)$ , then (4.14) trivially holds. Suppose that  $0 \notin F(x, p) + G(x)$ . Then, for every  $\varepsilon > 0$ , one can find  $w_\varepsilon \in F(x, p) + G(x)$  such that

$$\|w_\varepsilon\| < d(0, F(x, p) + G(x)) + \varepsilon. \quad (4.17)$$

We know that  $(x, p) \in B(\bar{x}, \tau) \times B(\bar{p}, t) \subset B(\bar{x}, \delta) \times B(\bar{p}, \delta)$ . Also, using (4.16), we have that  $d(0, F(x, p) + G(x)) < (m^{-1} - l)\rho$ , so for  $\varepsilon > 0$  sufficiently small,  $d(0, F(x, p) + G(x)) + \varepsilon < (m^{-1} - l)\rho < \delta$ . Consequently, we get from (4.17) that

$$0 \in B(w_\varepsilon, d(0, F(x, p) + G(x)) + \varepsilon) \subset B(w_\varepsilon, (m^{-1} - l)\rho) \subset B(w_\varepsilon, \delta), \quad (4.18)$$

so  $w_\varepsilon \in [F(x, p) + G(x)] \cap B(0, \delta)$ . Applying (i), one can find  $y_\varepsilon \in F(p, x) \cap B(\bar{y}, 2^{-1}s_1)$  and  $z_\varepsilon \in G(x) \cap B(-\bar{y}, 2^{-1}s_2)$  such that  $w_\varepsilon = y_\varepsilon + z_\varepsilon$ . Whence,  $B(y_\varepsilon, 2^{-1}s_1) \subset B(\bar{y}, s_1)$  and  $B(z_\varepsilon, 2^{-1}s_2) \subset B(-\bar{y}, s_2)$ .

Denote now  $r'_1 := 4^{-1}r_1$ ,  $s'_1 := 2^{-1}s_1$ ,  $r'_2 := 2^{-1}r_2$ ,  $s'_2 := 2^{-1}s_2$ . Summarizing,  $F_p^{-1}$  is  $l^{-1}$ -open at every  $(y', x') \in \text{Gr } F_p^{-1} \cap [B(y_\varepsilon, s'_1) \times B(x, r'_1)]$ ,  $G$  is  $m^{-1}$ -open at every  $(u', v') \in \text{Gr } G \cap [B(x, r'_2) \times B(z_\varepsilon, s'_2)]$  and  $l^{-1}m^{-1} > 1$ . We can apply then Theorem 3.3 for  $-G$ ,  $F_p^{-1}$ ,  $(x, -z_\varepsilon) \in \text{Gr}(-G)$ ,  $(y_\varepsilon, x) \in \text{Gr } F_p^{-1}$  and  $\rho_0 := (m^{-1} - l)^{-1}(d(0, F(x, p) + G(x)) + \varepsilon) < \rho < \min\{r'_1, r'_2, l^{-1}s'_1, ms'_2\}$  to obtain that

$$B(y_\varepsilon + z_\varepsilon, d(0, F(x, p) + G(x)) + \varepsilon) \subset (F_p + G)(B(x, \rho_0)).$$

Using (4.18), we obtain that  $0 \in (F_p + G)(B(x, \rho_0))$ , so there exists  $\tilde{x} \in B(x, \rho_0)$  such that  $0 \in F(\tilde{x}, p) + G(\tilde{x})$  or, equivalently,  $\tilde{x} \in S(p)$ . Hence

$$d(x, S(p)) \leq \|x - \tilde{x}\| < \rho_0 = d(0, F(x, p) + G(x)) + \varepsilon.$$

Making  $\varepsilon \rightarrow 0$ , we obtain (4.14).

For the final step of the proof, just observe that because  $F$  is Lipschitz-like with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), \bar{y})$  and  $G$  does not depend on  $p$ , we have that  $H$  defined by (4.12) is Lipschitz-like with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), 0)$ . Moreover,  $\widehat{\text{lip}}_p F((\bar{x}, \bar{p}), \bar{y}) = \widehat{\text{lip}}_p H((\bar{x}, \bar{p}), 0)$ . Then one can proceed as in the proof of the final part of Theorem 3.6 (i) to conclude the proof.  $\square$

The result would follow using just the local closedness of the graphs of  $F_p$  and  $G$  for  $p$  close to  $\bar{p}$  around appropriate points, but we preferred the actual formulation in order to avoid a more complicated assertion. Theorem 4.13 generalizes [1, Theorem 5.1 (ii)].

Note that, using Proposition 4.1, one could obtain on the basis of Artacho and Mordukhovich tools in [1] the converse links between the metric regularity/Lipschitz-like property of solution map  $S$  and the corresponding properties of the field map  $G$ .

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# OPENNESS STABILITY AND IMPLICIT MULTIFUNCTION THEOREMS. APPLICATIONS TO VARIATIONAL SYSTEMS

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**Abstract:** In this paper we aim to present two general results regarding, on one hand, the openness stability of set-valued maps and, on the other hand, the metric regularity behavior of the implicit multifunction related to a generalized variational system. Then, these results are applied in order to obtain, in a natural way, and in a widely studied case, several relations between the metric regularity moduli of the field maps defining the variational system and the solution map. Our approach allows us to complete and extend several very recent results in literature.

**Keywords:** set-valued mappings · linear openness · metric regularity · Lipschitz-like property · implicit multifunctions

**Mathematics Subject Classification (2010):** 90C30 · 49J53 · 54C60

## 1 Introduction

This paper belongs to the active area of research concerning parametric variational systems and it aims to enter into dialog with some very recent works of Aragón Artacho and Mordukhovich ([1], [2]) and Ngai, Tron and Théra ([13]). Note that, in turn, these papers extend many results of Dontchev and Rockafellar ([5], [6]).

Firstly, our research on the relations between metric regularity/Lipschitz moduli of an initial parametric field map and the associated implicit multifunction map led us to the rediscovery of a very nice Theorem of Ursescu [16] concerning the stability of openness of set-valued maps (Theorem 3.1 below). The proof we provide here for this result is appropriate enough for getting some extra assertions compared with the initial paper of Ursescu. Later in the paper, this result is a key ingredient in order to get a natural and precise answer to the question of how regularity constants of the involved maps relate each other.

Secondly, we were interested in enlarging the framework commonly used as being the defining form of a parametric variational system to the case of a general field map. To be more specific, let



$X, Y, P$  be Banach spaces,  $H : X \times P \rightrightarrows Y$  be a multifunction and define the implicit set-valued map  $S : P \rightrightarrows X$  by

$$S(p) = \{x \in X \mid 0 \in H(x, p)\}.$$

Then, in this second part of our work we find how metric regularity and Lipschitz properties of  $H$  and  $S$  are related under certain assumptions (Theorem 3.6). This second main result is in fact a general implicit multifunction theorem. After fixing these two main tools we are able to present in a natural way the situation where  $H$  is a sum of two set-valued maps  $F, G$  of the form  $H(x, p) = F(x, p) + G(x)$ . Note that this case is more general than the situation considered in [1] by the presence of the set-valued map  $F$  instead of a single-valued map. Moreover, this is the most general situation one can consider because it is not possible to get good results concerning the Lipschitz properties when  $G$  depends on the parameter  $p$ , in virtue of [1, Remark 3.6. (iii)]. However, note that in [13] the authors deal with a sort of metric regularity of the solution map associated to the sum of two parametric set-valued maps. In our framework, when we put at work together the two main results, we are in the position to indicate in a smooth manner the relations between the regularity moduli of  $S, G$  and  $F$ . We hope that our main results and their combination will bring more light on the previous results on this topic.

We would like to mention that, in comparison with [1] and [2], we get here only results in which the assumptions are on  $F$  and  $G$  and the conclusion concerns  $S$ . The converse situation considered in the quoted works (from  $S$  to  $G$ ) is not presented here because it has too many similarities with the corresponding results of Artacho and Mordukhovich. Any interested reader could find the arguments to obtain such results in our framework, but with Artacho and Mordukhovich tools.

The paper is organized as follows. In the next section we present the notations, the concepts and the basic facts we use in the sequel. The third section contains the main results of the paper we have presented in few words above. The last section investigates the widely studied form of the parametric variational systems we can find in literature. We show here how the main results concerning the stability of the linear openness of set-valued maps and the implicit multifunctions could be combined in order to get quite easily the estimation of the regularity constants of the solution map.

## 2 Preliminaries

This section contains some basic definitions and results used in the sequel. In what follows, we suppose that all the involved spaces are Banach. In this setting,  $B(x, r)$  and  $D(x, r)$  denote the open and the closed ball with center  $x$  and radius  $r$ , respectively. Sometimes we write  $\mathbb{D}_X$  for the closed unit ball of  $X$ . If  $x \in X$  and  $A \subset X$ , one defines the distance from  $x$  to  $A$  as  $d(x, A) := \inf\{\|x - a\| \mid a \in A\}$ . As usual, we use the convention  $d(x, \emptyset) = \infty$ . For a non-empty set  $A \subset X$  we put  $\text{cl } A$ ,  $\text{int } A$  for the topological closure and interior, respectively. Also, a set  $A$  is said to be locally closed if for every  $a \in A$ , there exists  $r > 0$  such that the set  $A \cap D(a, r)$  is closed. If  $\bar{a} \in A$  and  $A \cap V$  is locally closed, where  $V$  is a neighborhood of  $\bar{a}$ , we say that the set  $A$  is locally closed around  $\bar{a}$ .

Consider now a multifunction  $F : X \rightrightarrows Y$ . The domain and the graph of  $F$  are denoted respectively by

$$\text{Dom } F := \{x \in X \mid F(x) \neq \emptyset\}$$

and

$$\text{Gr } F = \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

If  $A \subset X$  then  $F(A) := \bigcup_{x \in A} F(x)$ . The inverse set-valued map of  $F$  is  $F^{-1} : Y \rightrightarrows X$  given by  $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$ .

Recall that a multifunction  $F$  is inner semicontinuous at  $(x, y) \in \text{Gr } F$  if for every open set  $D \subset Y$  with  $y \in D$ , there exists a neighborhood  $U \in \mathcal{V}(x)$  such that for every  $x' \in U$ ,  $F(x') \cap D \neq \emptyset$  (where  $\mathcal{V}(x)$  stands for the system of the neighborhoods of  $x$ ).

We remind now the concepts of openness at linear rate, metric regularity and Lipschitz-likeness of a multifunction around the reference point.

**Definition 2.1** Let  $L > 0$ ,  $F : X \rightrightarrows Y$  be a multifunction and  $(\bar{x}, \bar{y}) \in \text{Gr } F$ .

(i)  $F$  is said to be open at linear rate  $L > 0$ , or  $L$ -open around  $(\bar{x}, \bar{y})$  if there exist a positive number  $\varepsilon > 0$  and two neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{y})$  such that, for every  $\rho \in ]0, \varepsilon[$  and every  $(x, y) \in \text{Gr } F \cap [U \times V]$ ,

$$B(y, \rho L) \subset F(B(x, \rho)). \quad (2.1)$$

The supremum of  $L > 0$  over all the combinations  $(L, U, V, \varepsilon)$  for which (2.1) holds is denoted by  $\text{lop } F(\bar{x}, \bar{y})$  and is called the exact linear openness bound, or the exact covering bound of  $F$  around  $(\bar{x}, \bar{y})$ .

(ii)  $F$  is said to be Lipschitz-like, or has Aubin property around  $(\bar{x}, \bar{y})$  with constant  $L > 0$  if there exist two neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{y})$  such that, for every  $x, u \in U$ ,

$$F(x) \cap V \subset F(u) + L \|x - u\| \mathbb{D}_Y. \quad (2.2)$$

The infimum of  $L > 0$  over all the combinations  $(L, U, V)$  for which (2.2) holds is denoted by  $\text{lip } F(\bar{x}, \bar{y})$  and is called the exact Lipschitz bound of  $F$  around  $(\bar{x}, \bar{y})$ .

(iii)  $F$  is said to be metrically regular around  $(\bar{x}, \bar{y})$  with constant  $L > 0$  if there exist two neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{y})$  such that, for every  $(x, y) \in U \times V$ ,

$$d(x, F^{-1}(y)) \leq L d(y, F(x)). \quad (2.3)$$

The infimum of  $L > 0$  over all the combinations  $(L, U, V)$  for which (2.3) holds is denoted by  $\text{reg } F(\bar{x}, \bar{y})$  and is called the exact regularity bound of  $F$  around  $(\bar{x}, \bar{y})$ .

The next proposition contains the well-known links between the notions presented above. See [12, Theorems 1.49, 1.52] for more details about the proof.

**Proposition 2.2** Let  $F : X \rightrightarrows Y$  be a multifunction and  $(\bar{x}, \bar{y}) \in \text{Gr } F$ . Then  $F$  is open at linear rate around  $(\bar{x}, \bar{y})$  iff  $F^{-1}$  is Lipschitz-like around  $(\bar{y}, \bar{x})$  iff  $F$  is metrically regular around  $(\bar{x}, \bar{y})$ . Moreover, in every of the previous situations,

$$(\text{lop } F(\bar{x}, \bar{y}))^{-1} = \text{lip } F^{-1}(\bar{y}, \bar{x}) = \text{reg } F(\bar{x}, \bar{y}).$$

It is well known that the corresponding “at point” properties are significantly different from the “around point” ones. Let us introduce now some of these notions. For more related concepts we refer to [1].

**Definition 2.3** Let  $L > 0$ ,  $F : X \rightrightarrows Y$  be a multifunction and  $(\bar{x}, \bar{y}) \in \text{Gr } F$ .

(i)  $F$  is said to be open at linear rate  $L$ , or  $L$ -open at  $(\bar{x}, \bar{y})$  if there exists a positive number  $\varepsilon > 0$  such that, for every  $\rho \in ]0, \varepsilon[$ ,

$$B(\bar{y}, \rho L) \subset F(B(\bar{x}, \rho)). \quad (2.4)$$

The supremum of  $L > 0$  over all the combinations  $(L, \varepsilon)$  for which (2.4) holds is denoted by  $\text{plop } F(\bar{x}, \bar{y})$  and is called the exact punctual linear openness bound of  $F$  at  $(\bar{x}, \bar{y})$ .

(ii)  $F$  is said to be pseudocalm with constant  $L$ , or  $L$ -pseudocalm at  $(\bar{x}, \bar{y})$ , if there exists a neighborhood  $U \in \mathcal{V}(\bar{x})$  such that, for every  $x \in U$ ,

$$d(\bar{y}, F(x)) \leq L \|x - \bar{x}\|. \quad (2.5)$$

The infimum of  $L > 0$  over all the combinations  $(L, U)$  for which (2.5) holds is denoted by  $\text{psdclm } F(\bar{x}, \bar{y})$  and is called the exact bound of pseudocalmness for  $F$  at  $(\bar{x}, \bar{y})$ .

(iii)  $F$  is said to be metrically hemiregular with constant  $L$ , or  $L$ -metrically hemiregular at  $(\bar{x}, \bar{y})$  if there exists a neighborhood  $V \in \mathcal{V}(\bar{y})$  such that, for every  $y \in V$ ,

$$d(\bar{x}, F^{-1}(y)) \leq L \|y - \bar{y}\|. \quad (2.6)$$

The infimum of  $L > 0$  over all the combinations  $(L, V)$  for which (2.6) holds is denoted by  $\text{hemreg } F(\bar{x}, \bar{y})$  and is called the exact hemiregularity bound of  $F$  at  $(\bar{x}, \bar{y})$ .

The term of metric hemiregularity appears in [2, Definition 5.1], where the link with “Lipschitz lower semicontinuity” (i.e., pseudocalmness in our terminology) of the inverse multifunction is emphasized. The notion of pseudocalmness is used under the term of  $L$ -Lipschitz in [16], where other concepts of relative openness and relative  $L$ -Lipschitz properties are introduced and discussed.

The next proposition lists some equivalences between these “at point” notions. We give the (elementary) proof for the completeness.

**Proposition 2.4** *Let  $L > 0$ ,  $F : X \rightrightarrows Y$  and  $(\bar{x}, \bar{y}) \in \text{Gr } F$ . Then  $F$  is  $L$ -open at  $(\bar{x}, \bar{y})$  iff  $F^{-1}$  is  $L^{-1}$ -pseudocalm at  $(\bar{y}, \bar{x})$  iff  $F$  is  $L^{-1}$ -metrically hemiregular at  $(\bar{x}, \bar{y})$ . Moreover, in every of the previous situations,*

$$(\text{plop } F(\bar{x}, \bar{y}))^{-1} = \text{psdclm } F^{-1}(\bar{y}, \bar{x}) = \text{hemreg } F(\bar{x}, \bar{y}).$$

**Proof.** It's obvious from the very definitions that  $F^{-1}$  is  $L^{-1}$ -pseudocalm at  $(\bar{y}, \bar{x})$  iff  $F$  is  $L^{-1}$ -metrically hemiregular at  $(\bar{x}, \bar{y})$ . Suppose now that  $F$  is  $L$ -open at  $(\bar{x}, \bar{y})$ . Then there exists  $\varepsilon > 0$  such that, for every  $\rho \in ]0, \varepsilon[$ , (2.4) holds. Consider  $\varepsilon' := L\varepsilon$  and take arbitrarily  $y \in B(\bar{y}, \varepsilon')$ . Then there exist  $\rho \in ]0, \varepsilon[$  and  $\gamma$  arbitrary small such that  $\|y - \bar{y}\| = L\rho < L(\rho + \gamma) < L\varepsilon$ . Using the  $L$ -openness of  $F$  at  $(\bar{x}, \bar{y})$ ,  $y \in B(\bar{y}, L(\rho + \gamma)) \subset F(B(\bar{x}, \rho + \gamma))$ . Consequently, one can find  $x \in B(\bar{x}, \rho + \gamma) \cap F^{-1}(y)$ . Then, for  $\gamma$  arbitrary small,  $d(\bar{x}, F^{-1}(y)) \leq \|x - \bar{x}\| < \rho + \gamma$ , whence  $d(\bar{x}, F^{-1}(y)) \leq \rho = L^{-1} \|y - \bar{y}\|$ , and the first implication is now proved.

Suppose now that  $F^{-1}$  is  $L^{-1}$ -pseudocalm at  $(\bar{y}, \bar{x})$ , so there exists  $\varepsilon > 0$  such that, for every  $y \in B(\bar{y}, \varepsilon)$ ,  $d(\bar{x}, F^{-1}(y)) \leq L^{-1} \|y - \bar{y}\|$ . Take  $\varepsilon' := L^{-1}\varepsilon$  and arbitrarily  $\rho \in ]0, \varepsilon'[$ . If  $y' \in B(\bar{y}, L\rho)$ , we obtain that  $\|y' - \bar{y}\| < L\varepsilon' = \varepsilon$ , so  $d(\bar{x}, F^{-1}(y')) \leq L^{-1} \|y' - \bar{y}\| < \rho$ . Consequently, there exists  $x' \in F^{-1}(y')$  such that  $\|x' - \bar{x}\| < \rho$ , whence  $y' \in F(x') \subset F(B(\bar{x}, \rho))$ . The proof is now complete.  $\square$

See [17, Section 11] for an example of a multifunction which is open at linear rate at a point, hence on the basis of Proposition 2.4 is metrically hemiregular at this point, even if that multifunction is not metrically regular around any point.

Recall that  $\mathcal{L}(X, Y)$  denotes the normed vector space of linear bounded operators acting between  $X$  and  $Y$ . If  $A \in \mathcal{L}(X, Y)$ , then the “at” and “around point” notions do coincide. In fact,  $A$  is metrically regular around every  $x \in X$  iff  $A$  is metrically hemiregular at every  $x \in X$  iff  $A$  is open

with linear rate around every  $x \in X$  iff  $A$  is open with linear rate at every  $x \in X$  iff  $A$  is surjective. Moreover, in every of these cases we have

$$\text{hemreg } A = \text{reg } A = (\text{plop } A)^{-1} = (\text{lop } A)^{-1} = \|(A^*)^{-1}\|,$$

where  $A^* \in \mathcal{L}(Y^*, X^*)$  denotes the adjoint operator and  $\text{hemreg } A$ ,  $\text{reg } A$ ,  $\text{plop } A$  and  $\text{lop } A$  are common for all the points  $x \in X$  (see, for more details, [2, Proposition 5.2]).

In the following, we introduce the corresponding partial notions of linear openness, metric regularity and Lipschitz-like property around the reference point for a parametric set-valued map. Below, we denote by  $P$  the Banach space of parameters.

**Definition 2.5** Let  $L > 0$ ,  $F : X \times P \rightrightarrows Y$  be a multifunction,  $((\bar{x}, \bar{p}), \bar{y}) \in \text{Gr } F$  and for every  $p \in P$ , denote  $F_p(\cdot) := F(\cdot, p)$ .

(i)  $F$  is said to be open at linear rate  $L > 0$ , or  $L$ -open, with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), \bar{y})$  if there exist a positive number  $\varepsilon > 0$  and some neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{p})$ ,  $W \in \mathcal{V}(\bar{y})$  such that, for every  $\rho \in ]0, \varepsilon[$ , every  $p \in V$  and every  $(x, y) \in \text{Gr } F_p \cap [U \times W]$ ,

$$B(y, \rho L) \subset F_p(B(x, \rho)), \quad (2.7)$$

The supremum of  $L > 0$  over all the combinations  $(L, U, V, W, \varepsilon)$  for which (2.7) holds is denoted by  $\widehat{\text{lop}}_x F((\bar{x}, \bar{p}), \bar{y})$  and is called the exact linear openness bound, or the exact covering bound of  $F$  in  $x$  around  $((\bar{x}, \bar{p}), \bar{y})$ .

(ii)  $F$  is said to be Lipschitz-like, or has Aubin property, with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), \bar{y})$  with constant  $L > 0$  if there exist some neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{p})$ ,  $W \in \mathcal{V}(\bar{y})$  such that, for every  $x, u \in U$  and every  $p \in V$ ,

$$F_p(x) \cap W \subset F_p(u) + L \|x - u\| \mathbb{D}_Y. \quad (2.8)$$

The infimum of  $L > 0$  over all the combinations  $(L, U, V, W)$  for which (2.8) holds is denoted by  $\widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y})$  and is called the exact Lipschitz bound of  $F$  in  $x$  around  $((\bar{x}, \bar{p}), \bar{y})$ .

(iii)  $F$  is said to be metrically regular with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), \bar{y})$  with constant  $L > 0$  if there exist some neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{p})$ ,  $W \in \mathcal{V}(\bar{y})$  such that, for every  $(x, p, y) \in U \times V \times W$ ,

$$d(x, F_p^{-1}(y)) \leq L d(y, F_p(x)). \quad (2.9)$$

The infimum of  $L > 0$  over all the combinations  $(L, U, V, W)$  for which (2.9) holds is denoted by  $\widehat{\text{reg}}_x F((\bar{x}, \bar{p}), \bar{y})$  and is called the exact regularity bound of  $F$  in  $x$  around  $((\bar{x}, \bar{p}), \bar{y})$ .

Similarly, one can define the notions of linear openness, metric regularity and Lipschitz-like property with respect to  $p$  uniformly in  $x$ , and the corresponding exact bounds.

### 3 Main results

We begin our analysis with an interesting result due to Ursescu (see, e.g., [16, Theorem 1]), which brings into the light the key fact that the linear openness property of a difference type multifunction can be deduced from the corresponding linear openness properties of its terms. This result can be viewed as a deep generalization of the Graves Theorem (see the remark after Theorem 3.1). Different variants of this result were stated in [10], [3], and the more general assumptions are that

$Y$  is a linear metric space with shift invariant metric,  $X$  is metric space, and the graphs of the involved multifunctions are locally complete. Moreover, its importance seems to be crucial, because it can be putted into relation with a large number of (very) actual topics, as the strongly regular generalized equations of Robinson type (see [14]), allowing for the first time to deal with problems where both the terms are multivalued, but also with the inverse and implicit type theorems for set-valued mappings, with parametric variational inclusions and more, as one can see next. The proof we provide here is somehow similar with that of [16, Theorem 5], but it is more direct and simple. Besides the result itself, this new proof of it is a key element in extending the framework to the parametric case and, furthermore, to the rest of the applications we make precise in the last section.

**Theorem 3.1** *Let  $F : X \rightrightarrows Y$  and  $G : Y \rightrightarrows X$  be two multifunctions such that  $\text{Gr } F$  and  $\text{Gr } G$  are locally closed. Suppose that  $\text{Dom}(F - G^{-1})$  and  $\text{Dom}(G - F^{-1})$  are nonempty and let  $L > 0$  and  $M > 0$  be such that  $LM > 1$ . If  $F$  is  $L$ -open at every point of its graph, and  $G$  is  $M$ -open at every point of its graph, then  $F - G^{-1}$  is  $(L - M^{-1})$ -open at every point of its graph and  $G - F^{-1}$  is  $(M - L^{-1})$ -open at every point of its graph.*

**Proof.** We prove only the first assertion, the other one being completely symmetrical. Let  $(x, w) \in \text{Gr}(F - G^{-1})$ . Then there exist  $y \in F(x)$  and  $z \in G^{-1}(x)$  such that  $w = y - z$ . Define the multifunction  $(F, G^{-1}) : X \rightrightarrows Y \times Y$  by  $(F, G^{-1})(x) := F(x) \times G^{-1}(x)$  and remark that  $(x, y, z) \in \text{Gr}(F, G^{-1})$ . Because  $\text{Gr } F$  and  $\text{Gr } G$  are locally closed, it follows that  $\text{Gr}(F, G^{-1})$  is locally closed and one can find  $\rho > 0$  such that  $\text{Gr}(F, G^{-1}) \cap \text{cl } W$  is closed, where

$$W := B(x, \rho) \times B(y, L\rho) \times B(z, M^{-1}\rho). \quad (3.1)$$

Take  $u \in B(w, (L - M^{-1})\rho)$ . We must prove that  $u \in (F - G^{-1})(B(x, \rho))$ . One can find  $\tau \in ]0, 1[$  such that  $\|u - w\| < \tau(L - M^{-1})\rho$ . Endow the space  $X \times Y \times Y$  with the norm

$$\|(p, q, r)\|_0 := \tau(L - M^{-1}) \max\{\|p\|, L^{-1}\|q\|, M\|r\|\}$$

and apply the Ekeland variational principle to the function  $h : \text{Gr}(F, G^{-1}) \cap \text{cl } W \rightarrow \mathbb{R}_+$ ,

$$h(p, q, r) := \|u - (q - r)\|.$$

Then one can find a point  $(a, b, c) \in \text{Gr}(F, G^{-1}) \cap \text{cl } W$  such that

$$\|u - (b - c)\| \leq \|u - (y - z)\| - \|(a, b, c) - (x, y, z)\|_0 \quad (3.2)$$

and

$$\|u - (b - c)\| \leq \|u - (q - r)\| + \|(a, b, c) - (p, q, r)\|_0, \text{ for every } (p, q, r) \in \text{Gr}(F, G^{-1}) \cap \text{cl } W. \quad (3.3)$$

From (3.2) we have that

$$\begin{aligned} \tau(L - M^{-1}) \max\{\|a - x\|, L^{-1}\|b - y\|, M\|c - z\|\} &= \|(a, b, c) - (x, y, z)\|_0 \\ &\leq \|u - (y - z)\| = \|u - w\| \\ &< \tau(L - M^{-1})\rho, \end{aligned}$$

hence  $(a, b, c) \in W$ , and, in particular,  $a \in B(x, \rho)$ .

If  $u = b - c$ , then  $u \in (F - G^{-1})(a) \subset (F - G^{-1})(B(x, \rho))$  and the desired assertion is proved.

We want to show that  $u = b - c$  is the sole possible situation. For this, suppose by means of contradiction that  $u \neq b - c$ . Fix  $\varepsilon > 0$  such that  $L - \varepsilon > 0$  and define next

$$v := (L - \varepsilon) \|u - (b - c)\|^{-1} (u - (b - c)).$$

Then, for every  $s > 0$  sufficiently small, from the  $L$ -openness of  $F$  at  $(a, b) \in \text{Gr } F$ , we obtain that

$$b + sv \in B(b, Ls) \subset F(B(a, s)).$$

Consequently, there exists  $d \in B(a, s)$  such that  $b + sv \in F(d)$ . Obviously, one can find  $p$  with  $\|p\| < 1$  such that  $d = a + sp$ .

Also, using the  $M$ -openness of  $G$  at  $(c, a)$ , we have that for every  $s > 0$  sufficiently small,

$$B(a, s) \subset G(B(c, M^{-1}s)).$$

Hence, one can find  $e \in B(c, M^{-1}s)$  such that  $d \in G(e)$  or, equivalently,  $e \in G^{-1}(d)$ . Because we can write  $e = c + M^{-1}sq$  with  $\|q\| < 1$ , we finally have that  $(a + sp, b + sv, c + M^{-1}sq) \in \text{Gr}(F, G^{-1})$ . Taking a smaller  $s$  if necessary, we also have that  $(a + sp, b + sv, c + M^{-1}sq) \in W$ . We use now (3.3) to obtain that, for every  $s > 0$  sufficiently small,

$$\begin{aligned} \|u - (b - c)\| &\leq \|u - (b + sv - c - M^{-1}sq)\| + s \|(p, v, M^{-1}q)\|_0 \\ &\leq \|u - (b - c) - sv\| + M^{-1}s + s \|(p, v, M^{-1}q)\|_0. \end{aligned} \quad (3.4)$$

But

$$\|u - (b - c) - sv\| = \|\|u - (b - c)\| - s(L - \varepsilon)\|.$$

Eventually for even a smaller  $s$ , one obtains successively from (3.4) that

$$\begin{aligned} \|u - (b - c)\| &\leq \|u - (b - c)\| - s(L - \varepsilon) + M^{-1}s + s \|(p, v, M^{-1}q)\|_0, \\ L - M^{-1} - \varepsilon &\leq \tau(L - M^{-1}) \max\{\|p\|, L^{-1}\|v\|, M\|M^{-1}q\|\}, \\ L - M^{-1} - \varepsilon &\leq \tau(L - M^{-1}). \end{aligned}$$

Passing to the limit when  $\varepsilon \rightarrow 0$ , we get that  $1 \leq \tau$ , which is the contradiction. The proof is now complete.  $\square$

Let us point out that the previous theorem contains several results in literature, especially when  $F$  and  $G$  are single-valued (see, for more details, [16, p. 412]). Among these results, maybe the most famous one is that of Graves [11, p. 112], which can be easily deduced from Theorem 3.1 and Proposition 2.4. See, also, the comment after Corollary 3.4.

Notice that with a slight modification of the proof of Theorem 3.1, one obtains the next straightforward generalization (see [16, Theorem 3]).

**Theorem 3.2** *Let  $F : X \rightrightarrows Y$  and  $G_1, \dots, G_n : Y \rightrightarrows X$  be such that  $\text{Gr } F$  and  $\text{Gr } G_i$ ,  $i \in \{1, 2, \dots, n\}$ , are locally closed. Suppose  $\text{Dom}(F - G_1^{-1} - \dots - G_n^{-1})$  is nonempty and let  $L > 0$  and  $M_1, \dots, M_n > 0$  be such that  $L > M_1^{-1} + \dots + M_n^{-1}$ . If  $F$  is  $L$ -open at every point from a neighborhood of  $(x, y) \in \text{Gr } F$ , and  $G_i$  is  $M_i$ -open at every point from a neighborhood of  $(z_i, x) \in \text{Gr } G_i$  for every  $i \in \{1, 2, \dots, n\}$ , then  $F - G_1^{-1} - \dots - G_n^{-1}$  is  $(L - M_1^{-1} - \dots - M_n^{-1})$ -open at  $(x, y - z_1 - \dots - z_n)$ .*

**Proof.** Proceed as above, by taking the multifunction  $(F, G_1^{-1}, \dots, G_n^{-1}) : X \rightrightarrows Y^{n+1}$  given as

$$(F, G_1^{-1}, \dots, G_n^{-1})(x) := F(x) \times G_1^{-1}(x) \times \dots \times G_n^{-1}(x)$$

and observe that  $(x, y, z_1, \dots, z_n) \in \text{Gr}(F, G_1^{-1}, \dots, G_n^{-1})$ . The rest of the proof is more or less identical with that of Theorem 3.1, using now

$$W := B(x, \rho) \times B(y, L\rho) \times B(z_1, M_1^{-1}\rho) \times \dots \times B(z_n, M_n^{-1}\rho)$$

such that  $\text{Gr}(F, G_1^{-1}, \dots, G_n^{-1}) \cap \text{cl } W$  is closed, endowing the space  $X \times Y^{n+1}$  with the norm

$$\|(p, q, r_1, \dots, r_n)\|_0 := \tau(L - M_1^{-1} - \dots - M_n^{-1}) \max\{\|p\|, L^{-1}\|q\|, M_1\|r_1\|, \dots, M_n\|r_n\|\}$$

and applying the Ekeland variational principle to the function  $h : \text{Gr}(F, G_1^{-1}, \dots, G_n^{-1}) \cap \text{cl } W \rightarrow \mathbb{R}_+$ ,

$$h(p, q, r_1, \dots, r_n) := \|u - (q - r_1 - \dots - r_n)\|. \quad \square$$

The role of the next theorem is to precisely specify the constants involved in Theorem 3.1. This will be a key ingredient in the proof of several subsequent results.

**Theorem 3.3** *Let  $F : X \rightrightarrows Y$  and  $G : Y \rightrightarrows X$  be two multifunctions and  $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Y$  such that  $(\bar{x}, \bar{y}) \in \text{Gr } F$  and  $(\bar{z}, \bar{x}) \in \text{Gr } G$ . Suppose that the following assumptions are satisfied:*

- (i)  *$\text{Gr } F$  is locally closed around  $(\bar{x}, \bar{y})$ , so there exist  $\alpha_1, \beta_1 > 0$  such that  $\text{Gr } F \cap \text{cl}[B(\bar{x}, \alpha_1) \times B(\bar{y}, \beta_1)]$  is closed;*
- (ii)  *$\text{Gr } G$  is locally closed around  $(\bar{z}, \bar{x})$ , so there exist  $\alpha_2, \beta_2 > 0$  such that  $\text{Gr } G \cap \text{cl}[B(\bar{z}, \beta_2) \times B(\bar{x}, \alpha_2)]$  is closed;*
- (iii) *there exist  $L, r_1, s_1 > 0$  such that, for every  $(x', y') \in \text{Gr } F \cap [B(\bar{x}, r_1) \times B(\bar{y}, s_1)]$ ,  $F$  is  $L$ -open at  $(x', y')$ ;*
- (iv) *there exist  $M, r_2, s_2 > 0$  such that, for every  $(v', u') \in \text{Gr } G \cap [B(\bar{z}, s_2) \times B(\bar{x}, r_2)]$ ,  $G$  is  $M$ -open at  $(v', u')$ ;*
- (v)  *$LM > 1$ .*

*Then for every  $\rho \in ]0, \varepsilon[$ , where  $\varepsilon := \min\{\alpha_1, \alpha_2, L^{-1}\beta_1, M\beta_2, r_1, r_2, L^{-1}s_1, Ms_2\}$ ,*

$$B(\bar{y} - \bar{z}, (L - M^{-1})\rho) \subset (F - G^{-1})(B(\bar{x}, \rho)).$$

*Moreover, for every  $\rho \in (0, 2^{-1}\varepsilon)$  and every  $(x, y, z) \in \text{Gr}(F, G^{-1}) \cap [B(\bar{x}, 2^{-1}\varepsilon) \times B(\bar{y}, 2^{-1}\varepsilon) \times B(\bar{z}, 2^{-1}\varepsilon)]$ ,*

$$B(y - z, (L - M^{-1})\rho) \subset (F - G^{-1})(B(x, \rho)).$$

**Proof.** We only sketch the proof, pointing out the differences with respect to the proof of Theorem 3.1.

For the first part, take  $\rho \in ]0, \varepsilon[$ , define, as above, the multifunction  $(F, G^{-1})$ , and observe that the choice of  $\varepsilon$  implies that  $\text{Gr}(F, G^{-1}) \cap \text{cl } W$  is closed, where  $W := B(\bar{x}, \rho) \times B(\bar{y}, L\rho) \times B(\bar{z}, M^{-1}\rho)$ .

Take again  $u \in B(\bar{y} - \bar{z}, (L - M^{-1})\rho)$  and follow the same steps as above to obtain that  $(a, b, c) \in W$ . We only need to know that  $F$  is  $L$ -open at  $(a, b)$  and that  $G$  is  $M$ -open at  $(c, a)$  to complete the proof, but this follows again from the choice of  $\varepsilon$ .

For the second part, we define  $W_1 := B(x, \rho) \times B(y, L\rho) \times B(z, M^{-1}\rho)$  and we remark again that  $\text{Gr}(F, G^{-1}) \cap \text{cl } W_1$  is closed, because  $W_1 \subset B(\bar{x}, \min\{\alpha_1, \alpha_2\}) \times B(\bar{y}, \beta_1) \times B(\bar{z}, \beta_2)$ . The rest



of the proof is the same as above, observing only that because  $(a, b, c) \in W_1 \subset B(\bar{x}, \min\{r_1, r_2\}) \times B(\bar{y}, s_1) \times B(\bar{z}, s_2)$ ,  $F$  is  $L$ -open at  $(a, b)$  and  $G$  is  $M$ -open at  $(c, a)$ .  $\square$

We want to emphasize that if  $F$  is  $L$ -open around  $(\bar{x}, \bar{y})$ , then  $F$  satisfies the property from the third item, and a similar observation is valid for  $G$ . Also, if one of the two multifunctions which appear in the previous result is univoque, then one can obtain the openness around the reference point of the difference.

**Corollary 3.4** *Let  $f : X \rightarrow Y$  be a function,  $G : Y \rightrightarrows X$  be a multifunction,  $L, M > 0$  and  $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Y$  such that  $\bar{y} = f(\bar{x})$  and  $(\bar{z}, \bar{x}) \in \text{Gr } G$ . Suppose that the following assumptions are satisfied:*

- (i)  $f$  is Lipschitz continuous around  $\bar{x}$ ;
  - (ii)  $\text{Gr } G$  is locally closed around  $(\bar{z}, \bar{x})$ ;
  - (iii)  $f$  is  $L$ -open around  $(\bar{x}, \bar{y})$ ;
  - (iv)  $G$  is  $M$ -open around  $(\bar{z}, \bar{x})$ ;
  - (v)  $LM > 1$ .
- Then  $f - G^{-1}$  is  $(L - M^{-1})$ -open around  $(\bar{x}, \bar{y} - \bar{z})$ .*

**Proof.** From the Lipschitz property, we obtain the local closedness of  $\text{Gr } f$  around  $(\bar{x}, \bar{y})$ , so one deduces a similar assertion as in (i) of the previous Corollary. Observe also that the rest of the assumptions are the same or stronger compared to the previous result, so we suppose in the following that the conditions are formulated using the same constants as above. Define again  $\varepsilon := \min\{\alpha_1, \alpha_2, L^{-1}\beta_1, M\beta_2, r_1, r_2, L^{-1}s_1, Ms_2\}$ ,  $\gamma := \min\{4^{-1}\varepsilon, (4l)^{-1}\varepsilon\}$ , where  $l > 0$  is the Lipschitz constant from (i). Take now  $\rho \in ]0, 2^{-1}\varepsilon[$  and  $(x, v) \in \text{Gr}(f - G^{-1}) \cap [B(\bar{x}, \gamma) \times B(\bar{y} - \bar{z}, \gamma)]$ , so there exist  $y = f(x)$  and  $z \in G^{-1}(x)$  such that  $v = y - z$ . Then we get that

$$\begin{aligned} x &\in B(\bar{x}, \gamma) \subset B(\bar{x}, 2^{-1}\varepsilon), \\ y &\in B(\bar{y}, l\gamma) \subset B(\bar{y}, 4^{-1}\varepsilon) \subset B(\bar{y}, 2^{-1}\varepsilon), \\ z &\in \bar{z} + (y - \bar{y}) + B(0, \gamma) \subset \bar{z} + B(0, 4^{-1}\varepsilon) + B(0, 4^{-1}\varepsilon) \subset B(\bar{z}, 2^{-1}\varepsilon), \end{aligned}$$

where we used (i) in the second row of inclusion.

Consequently,  $(x, y, z) \in \text{Gr}(f, G^{-1}) \cap [B(\bar{x}, 2^{-1}\varepsilon) \times B(\bar{y}, 2^{-1}\varepsilon) \times B(\bar{z}, 2^{-1}\varepsilon)]$ , so using the final part of the previous result we know that  $B(y - z, (L - M^{-1})\rho) \subset (f - G^{-1})(B(x, \rho))$ , and the proof is complete.  $\square$

As an easy consequence of the previous corollary, one can obtain the celebrated result of Lyusternik-Graves [11, p. 112], taking  $f$  as the Fréchet differential at a point of a continuously Fréchet differentiable function  $h$  and  $G := (h - f)^{-1}$ .

Another remark is that when one denotes the set-valued map  $-G^{-1}$  by  $F$  in the previous corollary, then one gets (in our specific framework) the so-called extended Lyusternik-Graves theorem (see Theorem 1 and the related comments from [4]).

The next corollary, which can be seen as a parametric version of a result of Graves, is the same with [2, Proposition 3.2] and gives sufficient conditions for the partial metric regularity of a function, but here is obtained as an easy consequence of Corollary 3.4.

**Corollary 3.5** *Let  $f : X \times P \rightarrow Y$  be a function between Banach spaces which is continuous around  $(\bar{x}, \bar{p})$  and let  $A \in \mathcal{L}(P, Y)$  be a surjective linear operator such that there exists  $\alpha > 0$  such*

that  $\text{lop } A > \alpha$  and for every  $p, p'$  in a neighborhood  $U$  of  $\bar{p}$  and every  $x$  in a neighborhood  $V$  of  $\bar{x}$ ,

$$\|f(x, p) - f(x, p') - A(p - p')\| \leq \alpha \|p - p'\|.$$

Then  $f$  is open with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), f(\bar{x}, \bar{p}))$  with

$$\widehat{\text{lop}}_p f(\bar{x}, \bar{p}) \geq \text{lop } A - \alpha.$$

Equivalently,  $f$  is metrically regular with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), f(\bar{x}, \bar{p}))$  with

$$\widehat{\text{reg}}_p f(\bar{x}, \bar{p}) \leq \frac{\text{reg } A}{1 - \alpha \cdot \text{reg } A}.$$

**Proof.** Just take  $x \in V$  and apply Corollary 3.4 for  $f := -A$  and  $G := (f(x, \cdot) - A)^{-1}$ . Remark that the constants  $\varepsilon$  and  $\gamma$  from the previous proof are the same for every  $x \in V$ .  $\square$

Note first that the constants in right-hand sides of the relations from the conclusion are coming directly from the openness result, as an easy consequence. Another remark concerns the fact that in the case where  $f$  is (strictly) partially differentiable with respect to  $p$ , then, as it is often the case in literature, one can take the partial differential with respect to  $p$  instead of  $A$ , in which case  $\widehat{\text{reg}}_p f(\bar{x}, \bar{p}) = \|(\nabla_p f(\bar{x}, \bar{p})^*)^{-1}\|$  (see [14], [1], [2, Proposition 3.4]).

Here comes the second main result of the paper. For this, we use some ideas we have previously developed in [8]. Let us introduce the objects we deal with. Remind that, for a multifunction  $H : X \times P \rightrightarrows Y$ , we can define the implicit set-valued map  $S : P \rightrightarrows X$  by:

$$S(p) = \{x \in X \mid 0 \in H(x, p)\}.$$

Note that a more general solution map (see [8], [13]) could be investigated from the point of view of several metric regularity concepts. We prefer the use of  $S$  in the present form for clarity and unity of the results.

The result we present is an implicit multifunction theorem and shows some interesting interrelations between the partial openness with respect to a variable plus the Lipschitz-like property with respect to the other variable of the original multifunction, and the Lipschitz-like, or the metric regularity of the implicit multifunction, respectively.

**Theorem 3.6** *Let  $X, P$  be metric spaces,  $Y$  be a normed vector space,  $H : X \times P \rightrightarrows Y$  be a set-valued map and  $(\bar{x}, \bar{p}, 0) \in \text{Gr } H$ . Denote by  $H_p(\cdot) := H(\cdot, p)$ ,  $H_x(\cdot) := H(x, \cdot)$ .*

*(i) If  $H$  is open at linear rate  $c > 0$  with respect to  $x$  uniformly in  $p$  around  $(\bar{x}, \bar{p}, 0)$ , then there exist  $\alpha, \beta, \gamma > 0$  such that, for every  $(x, p) \in B(\bar{x}, \alpha) \times B(\bar{p}, \beta)$ ,*

$$d(x, S(p)) \leq c^{-1} d(0, H(x, p) \cap B(0, \gamma)). \quad (3.5)$$

*If, moreover,  $H$  is inner semicontinuous at  $(\bar{x}, \bar{p}, 0)$ , then there exist  $\alpha', \beta' > 0$  such that, for every  $(x, p) \in B(\bar{x}, \alpha') \times B(\bar{p}, \beta')$ ,*

$$d(x, S(p)) \leq c^{-1} d(0, H(x, p)). \quad (3.6)$$

*Without the inner semicontinuity assumption on  $H$ , suppose, in addition, that  $H$  is Lipschitz-like with respect to  $p$  uniformly in  $x$  around  $(\bar{x}, \bar{p}, 0)$ . Then  $S$  is Lipschitz-like around  $(\bar{p}, \bar{x})$  and*

$$\text{lip } S(\bar{p}, \bar{x}) \leq c^{-1} \widehat{\text{lip}}_p H((\bar{x}, \bar{p}), 0). \quad (3.7)$$

(ii) If  $H$  is open at linear rate  $c > 0$  with respect to  $p$  uniformly in  $x$  around  $(\bar{x}, \bar{p}, 0)$ , then there exist  $\alpha, \beta, \gamma > 0$  such that, for every  $(x, p) \in B(\bar{x}, \alpha) \times B(\bar{p}, \beta)$ ,

$$d(p, S^{-1}(x)) \leq c^{-1}d(0, H(x, p) \cap B(0, \gamma)). \quad (3.8)$$

Either of the following assertions are independent:

If, moreover,  $H$  is inner semicontinuous at  $(\bar{x}, \bar{p}, 0)$ , then there exist  $\alpha', \beta' > 0$  such that, for every  $(x, p) \in B(\bar{x}, \alpha') \times B(\bar{p}, \beta')$ ,

$$d(p, S^{-1}(x)) \leq c^{-1}d(0, H(x, p)). \quad (3.9)$$

Without the inner semicontinuity assumption on  $H$ , suppose, in addition, that  $H$  is Lipschitz-like with respect to  $x$  uniformly in  $p$  around  $(\bar{x}, \bar{p}, 0)$ . Then  $S$  is metrically regular around  $(\bar{p}, \bar{x})$  and

$$\text{reg } S(\bar{p}, \bar{x}) \leq c^{-1} \widehat{\text{lip}}_x H((\bar{x}, \bar{p}), 0). \quad (3.10)$$

**Proof.** We will prove only the first item, because for the second one it suffices to observe that, defining the multifunction  $T := S^{-1}$ , the proof is completely symmetrical, using  $T$  instead of  $S$ . Moreover, using Proposition 2.2, we know that  $\text{reg } S(\bar{p}, \bar{x}) = \text{lip } T(\bar{x}, \bar{p})$  and then (3.10) follows from (3.7).

For the (i) item, we know that there exist  $r, s, t, c, \varepsilon > 0$  such that, for every  $\rho \in (0, \varepsilon)$ , every  $p \in B(\bar{p}, t)$  and every  $(x, y) \in \text{Gr } H_p \cap [B(\bar{x}, r) \times B(0, s)]$ ,

$$B(y, c\rho) \subset H_p(B(x, \rho)).$$

Take now  $\rho \in ]0, \min\{\varepsilon, c^{-1}s\}[$ . Set  $\alpha := r$ ,  $\beta := t$ ,  $\gamma := c\rho$  and fix arbitrary  $(x, p) \in B(\bar{x}, \alpha) \times B(\bar{p}, \beta)$ . If  $H(x, p) \cap B(0, c\rho) = \emptyset$ , then  $d(0, H(x, p) \cap B(0, \gamma)) = +\infty$  and (3.5) trivially holds. Suppose next that  $H(x, p) \cap B(0, c\rho) \neq \emptyset$ . If  $0 \in H(x, p)$ , then  $0 \in H(x, p) \cap B(0, c\rho)$ , and, again, (3.5) trivially holds. Suppose now that  $0 \notin H(x, p) \cap B(0, c\rho)$ . Then for every  $\xi > 0$ , there exists  $y_\xi \in H(x, p) \cap B(0, c\rho)$  such that

$$\|y_\xi\| < d(0, H(x, p) \cap B(0, c\rho)) + \xi.$$

Because  $d(0, H(x, p) \cap B(0, c\rho)) < c\rho$ , we can choose  $\xi$  sufficiently small such that  $d(0, H(x, p) \cap B(0, c\rho)) + \xi < c\rho$ . Consequently,

$$0 \in B(y_\xi, d(0, H(x, p) \cap B(0, c\rho)) + \xi) \subset B(y_\xi, c\rho). \quad (3.11)$$

Observe now that  $x \in B(\bar{x}, r)$ ,  $p \in B(\bar{p}, t)$ ,  $y_\xi \in B(0, d(0, H(x, p) \cap B(0, c\rho)) + \xi) \subset B(0, c\rho) \subset B(0, s)$ ,  $y_\xi \in H(x, p)$  and denote  $\rho_0 := c^{-1}(d(0, H(x, p) \cap B(0, c\rho)) + \xi) < \rho < \varepsilon$ .

But we know that

$$B(y_\xi, c\rho_0) \subset H_p(B(x, \rho_0)),$$

hence, using also (3.11), one obtains that there exists  $x_0 \in B(x, \rho_0)$  such that  $0 \in H(x_0, p)$ , which is equivalent to  $x_0 \in S(p)$ . Then

$$d(x, S(p)) \leq d(x, x_0) < \rho_0 = c^{-1}(d(0, H(x, p) \cap B(0, c\rho)) + \xi).$$

Making  $\xi \rightarrow 0$ , we obtain (3.5).

In the case that  $H$  is inner semicontinuous at  $(\bar{x}, \bar{p}, 0)$ , one can find  $\delta_1, \delta_2 > 0$  such that, for every  $(x, p) \in B(\bar{x}, \delta_1) \times B(\bar{p}, \delta_2)$ ,

$$H(x, p) \cap B(0, c\rho) \neq \emptyset. \quad (3.12)$$

Then  $\alpha' := \min\{r, \gamma\}$ ,  $\beta' := \min\{t, \delta\}$  are appropriate such that (3.6) is satisfied.

Suppose now that  $H$  is Lipschitz-like with respect to  $p$  uniformly in  $x$  around  $(\bar{x}, \bar{p}, 0)$ . Then there exist  $l, a, b, \tau > 0$  such that  $\tau < c\rho$  and for every  $x \in B(\bar{x}, a)$  and every  $p_1, p_2 \in B(\bar{p}, b)$ ,

$$H(x, p_1) \cap D(0, \tau) \subset H(x, p_2) + ld(p_1, p_2)\mathbb{D}_Y. \quad (3.13)$$

Take  $\bar{\alpha} := \min\{a, \alpha\}$ ,  $\bar{\beta} = \min\{b, \beta, (2l)^{-1}\tau\}$ ,  $p_1, p_2 \in B(\bar{p}, \bar{\beta})$  and  $x \in S(p_1) \cap B(\bar{x}, \bar{\alpha})$ . Then  $0 \in H(x, p_1) \cap D(0, \tau)$ , whence, using (3.13), there exists  $y'' \in \mathbb{D}_Y$  such that  $y' := l \cdot d(p_1, p_2)y'' \in H(x, p_2)$  with  $\|y'\| \leq l \cdot [d(p_1, \bar{p}) + d(\bar{p}, p_2)] \leq \tau < c\rho$ . Hence,  $y' \in H(x, p_2) \cap B(0, \gamma)$ , so using (3.5), we get that

$$d(x, S(p_2)) \leq c^{-1}d(0, H(x, p_2) \cap B(0, \gamma)) \leq c^{-1}\|y'\| \leq c^{-1}ld(p_1, p_2).$$

Consequently, because  $l$  can be chosen arbitrarily close to  $\widehat{\text{lip}}_p H((\bar{x}, \bar{p}), 0)$ , it follows that  $S$  is Lipschitz-like around  $(\bar{p}, \bar{x})$  and  $\text{lip } S(\bar{p}, \bar{x}) \leq c^{-1}\widehat{\text{lip}}_p H((\bar{x}, \bar{p}), 0)$ . The proof is now complete.  $\square$

It is worth to be mentioned that one can obtain in the final parts of the two items of the above theorem even a kind of graphical regularity, following the technique from [8, Theorem 5.2] (see, also, [9]).

## 4 Applications

This section is dedicated to the investigation of the case where the mapping  $H$  is given as a sum of two set-valued maps in the sense we shall precise. However, we start with an application of Theorem 3.6 and we get an implicit multifunction result which generalizes [2, Theorem 3.5] to the case where the set-valued map  $\Gamma$  is constructed using the sum between a function and a multifunction. Also, in the virtue of Corollary 3.5, we can conclude that this result generalizes also [1, Lemma 3.1].

**Proposition 4.1** *Let  $X, Y, Z, W$  be Banach spaces,  $F : X \times Y \rightrightarrows Z$  be a multifunction,  $g : W \rightarrow Z$  be a function and  $(\bar{x}, \bar{y}, \bar{z}, \bar{w}) \in X \times Y \times Z \times W$  be such that  $\bar{z} := -g(\bar{w}) \in F(\bar{x}, \bar{y})$ . Consider the implicit multifunction  $\Gamma : Y \times W \rightrightarrows X$  defined by*

$$\Gamma(y, w) := \{x \in X \mid 0 \in F(x, y) + g(w)\}.$$

*Suppose that the following conditions are satisfied:*

- (i)  $F$  is Lipschitz-like with respect to  $y$  uniformly in  $x$  around  $((\bar{x}, \bar{y}), \bar{z})$  with constant  $\eta \geq 0$ ;
- (ii)  $F$  is metrically regular with respect to  $x$  uniformly in  $y$  around  $((\bar{x}, \bar{y}), \bar{z})$  with constant  $k > 0$ ;
- (iii)  $g$  is locally Lipschitzian around  $\bar{w}$  with constant  $\lambda > 0$ .

*Then there exists  $\alpha > 0$  such that for every  $(y, w), (y', w') \in D(\bar{y}, \alpha) \times D(\bar{w}, \alpha)$  and for every  $\varepsilon > 0$ ,*

$$\Gamma(y', w') \cap D(\bar{x}, \alpha) \subset \Gamma(y, w) + (k + \varepsilon)(\eta\|y - y'\| + \lambda\|w - w'\|)\mathbb{D}_X.$$

*In particular,  $\Gamma$  is Lipschitz-like around  $((\bar{y}, \bar{w}), \bar{x})$  with the following estimate*

$$\text{lip } \Gamma((\bar{y}, \bar{w}), \bar{x}) \leq \widehat{\text{reg}}_x F((\bar{x}, \bar{y}), \bar{z}) \cdot \max\{\widehat{\text{lip}}_y F(\bar{x}, \bar{y}, \bar{z}), \text{lip } g(\bar{w})\}.$$

**Proof.** Define  $P := Y \times W$  and  $H : X \times P \rightrightarrows Z$  by  $H(x, (y, w)) := F(x, y) + g(w)$ . Observe also that

$$\Gamma(y, w) = \{x \in X \mid 0 \in H(x, (y, w))\},$$

and denote  $H_{(y,w)}(\cdot) := H(\cdot, (y, w))$ . For every  $y$  close to  $\bar{y}$  we know from (ii) and Proposition 2.2 that  $F_y$  is  $k^{-1}$ -open at points from its graph around  $(\bar{x}, \bar{z})$ . Therefore, we can conclude that there exists  $\varepsilon > 0$  such that for every  $\rho \in ]0, \varepsilon[$ , every  $y \in B(\bar{y}, \varepsilon)$  and every  $(x, z') \in \text{Gr } F_y \cap [B(\bar{x}, \varepsilon) \times B(\bar{z}, \varepsilon)]$ ,

$$B(z', k^{-1}\rho) \subset F_y(B(x, \rho)). \quad (4.1)$$

Take  $(y, w) \in B(\bar{y}, \varepsilon) \times B(\bar{w}, 2^{-1}\lambda^{-1}\varepsilon)$  and  $(x, z) \in \text{Gr } H_{(y,w)} \cap [B(\bar{x}, \varepsilon) \times B(0, 2^{-1}\varepsilon)]$ , and  $z' := z - g(w)$ . Then, using (iii), we have that

$$\|z' - \bar{z}\| \leq \|z\| + \|\bar{z} + g(w)\| < 2^{-1}\varepsilon + \lambda\|w - \bar{w}\| < \varepsilon,$$

so one can use (4.1) to prove that

$$B(z, k^{-1}\rho) = B(z', k^{-1}\rho) - g(w) \subset F_y(B(x, \rho)) - g(w) = H_{(y,w)}(B(x, \rho)).$$

But this shows, applying Theorem 3.6, that there exist  $\beta > 0$  such that, for every  $(y, z, w) \in B(\bar{y}, \beta) \times B(\bar{z}, \beta) \times B(\bar{w}, \beta)$ ,

$$d(y, \Gamma(y, w)) \leq kd(0, H(x, (y, w)) \cap B(0, \beta)). \quad (4.2)$$

We want to prove that there exists  $c > 0$  such that  $(\eta + 1)c < \beta$ , for every  $x \in B(\bar{x}, c)$ ,  $(y, w), (y', w') \in B(\bar{y}, c) \times B(\bar{w}, c)$ ,

$$H(x, (y, w)) \cap D(0, c) \subset H(x, (y', w')) + (\eta\|y - y'\| + \lambda\|w - w'\|)\mathbb{D}_Z. \quad (4.3)$$

In particular, we will prove that  $H$  is Lipschitz-like with respect to  $(y, w)$  uniformly in  $x$  around  $(\bar{x}, (\bar{y}, \bar{z}), 0)$ .

Because of (i), we know that there exists  $a > 0$  such that for every  $x \in B(\bar{x}, a)$  and every  $y, y' \in B(\bar{y}, a)$ ,

$$F(x, y) \cap D(\bar{z}, a) \subset F(x, y') + \eta\|y - y'\|\mathbb{D}_Z. \quad (4.4)$$

Also, because of (iii), we can find  $b > 0$  such that for every  $w, w' \in B(\bar{w}, b)$ ,

$$g(w) \in g(w') + \lambda\|w - w'\|\mathbb{D}_Z. \quad (4.5)$$

Choose now  $c \in ]0, \min\{(\lambda+1)^{-1}a, b, (\eta+\lambda)^{-1}\beta\}[$  and take arbitrary  $x \in B(\bar{x}, c)$ ,  $(y, w), (y', w') \in B(\bar{y}, c) \times B(\bar{w}, c)$ . Furthermore, choose  $z \in H(x, (y, w)) \cap D(0, c)$ . Then  $z - g(w) \in F(x, y)$  and because of (4.5), we know that  $-g(w) \in D(\bar{z}, \lambda\|w - \bar{w}\|) \subset B(\bar{z}, \lambda c)$ , whence  $z - g(w) \in B(\bar{z}, (\lambda + 1)c) \subset B(\bar{z}, a)$ . One can use now (4.4) to obtain that  $z - g(w) \in F(x, y') + \eta\|y - y'\|\mathbb{D}_Z$ . Adding with (4.5), one finally gets (4.3).

Take now  $\gamma \in ]0, \min\{\beta, 2^{-1}c\}[$ ,  $(y, w), (y', w') \in B(\bar{y}, \gamma) \times B(\bar{w}, \gamma)$ ,  $x \in \Gamma(y, w) \cap D(\bar{x}, \gamma)$  and arbitrary  $\varepsilon > 0$ . Hence,  $0 \in H(x, (y, w)) \cap D(0, c)$ . Then, using (4.3), we have that there exists  $u \in \mathbb{D}_Z$  such that  $(\eta\|y - y'\| + \lambda\|w - w'\|)u \in H(x, (y', w')) \cap B(0, \beta)$  (because  $(\eta + \lambda)c < \beta$ ). Finally, using (4.2), we get that

$$\begin{aligned} d(x, \Gamma(y', w')) &\leq kd(0, H(x, (y', w')) \cap B(0, \beta)) \leq k(\eta\|y - y'\| + \lambda\|w - w'\|) \\ &< (k + \varepsilon)(\eta\|y - y'\| + \lambda\|w - w'\|), \end{aligned}$$

which completes the proof.  $\square$

Note that the final conclusion of Theorem 3.6 (i), but the estimations (3.5) and (3.6), could be obtained as a consequence of Proposition 4.1, taking  $g \equiv 0$ . We note as well that for  $W := Z$  and  $g(z) := -z$  for every  $z$  we can get an even more general implicit multifunction result (see [8, Theorem 5.2]).

The next technical notion will be used in the sequel, mainly to prove a Lipschitz-like property of the sum between two multifunctions. In this way (in contrast to [13, Corollary 18]), we avoid the strong requirements of the single-valuedness and full Lipschitz property of the field map  $G$  at the reference point.

**Definition 4.2** Let  $F : X \rightrightarrows Y$ ,  $G : X \rightrightarrows Y$  be two multifunctions and  $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Y$  such that  $\bar{y} \in F(\bar{x})$ ,  $\bar{z} \in G(\bar{x})$ . We say that the multifunction  $(F, G)$  is locally sum-stable around  $(\bar{x}, \bar{y}, \bar{z})$  if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that, for every  $x \in B(\bar{x}, \delta)$  and every  $w \in (F + G)(x) \cap B(\bar{y} + \bar{z}, \delta)$ , there exist  $y \in F(x) \cap B(\bar{y}, \varepsilon)$  and  $z \in G(x) \cap B(\bar{z}, \varepsilon)$  such that  $w = y + z$ .

This definition is illustrated, at a first glance, by two simple examples. First example displays a simple situation where this condition holds true. Note that  $\mathbb{R}$  and  $\mathbb{Q}$  denote the fields of reals and rationals, respectively.

**Example 4.3** Let  $F : \mathbb{R} \rightrightarrows \mathbb{R}$  given, for any  $x \in \mathbb{R}$ , by

$$F(x) := ]0, |x| [.$$

Take now  $G := F$ . We will prove that  $(F, G)$  is locally sum-stable around any  $(\bar{x}, \bar{y}, \bar{z}) \in \text{Gr}(F, G)$ . In order to prove this, let us take, without loosing of generality,  $\bar{x} > 0$  and  $\bar{y}, \bar{z} \in ]0, \bar{x}[$ . Then fix  $\varepsilon > 0$  and choose  $\delta \in ]0, \min\{2^{-1}(\bar{x} - \bar{y}), 2^{-1}(\bar{x} - \bar{z}), \varepsilon, \bar{y}, \bar{z}\}[$ . Further, choose  $x \in ]\bar{x} - \delta, \bar{x} + \delta[$  and  $w \in (F + G)(x) \cap B(\bar{y} + \bar{z}, \delta) = ]0, 2x[ \cap ]\bar{y} + \bar{z} - \delta, \bar{y} + \bar{z} + \delta[$ . Now, if  $w < \bar{y} + \bar{z}$ , then there exist  $\gamma \in ]0, \delta[$  s.t.  $w = \bar{y} + \bar{z} - \gamma$ . Consider  $y := \bar{y} - 2^{-1}\gamma$  and  $z = \bar{z} - 2^{-1}\gamma$ . Then,  $\bar{y} - y = 2^{-1}\gamma < \delta < \varepsilon$ , and  $y > \bar{y} - \delta > 0$ . Moreover,  $y = \bar{y} - 2^{-1}\gamma < \bar{x} - 2\delta < x$ , i.e.  $y \in F(x)$ . Similarly for  $z$ . If  $w > \bar{y} + \bar{z}$ , then there exist  $\gamma \in ]0, \delta[$  s.t.  $w = \bar{y} + \bar{z} + \gamma$ . Consider  $y := \bar{y} + 2^{-1}\gamma$  and  $z = \bar{z} + 2^{-1}\gamma$ . Of course,  $y > 0$  and  $y - \bar{y} = 2^{-1}\gamma < \delta < \varepsilon$ . On the other hand,  $y = \bar{y} + 2^{-1}\gamma < \bar{y} + \delta < \bar{x} - \delta < x$ , i.e.  $y \in F(x)$ . The same calculation holds for  $z$  and the proof is complete.

The next example describes a situation where the sum-stable condition does not hold.

**Example 4.4** Let  $F : \mathbb{R} \rightrightarrows \mathbb{R}$  be given by

$$F(x) := \begin{cases} [-1, 1], & \text{if } x \in \mathbb{R} \setminus \{1\} \\ 2, & \text{if } x = 1 \end{cases}$$

and  $G : \mathbb{R} \rightrightarrows \mathbb{R}$  given, for any  $x \in \mathbb{R}$ , by  $G(x) := -F(x)$ . Then one can easily see that  $(F, G)$  is not sum-stable at  $(\bar{x}, \bar{y}, \bar{z}) := (1, 2, -2)$  because  $0 \in (F + G)(x)$  for every  $x \in \mathbb{R}$  but if  $x \neq 1$ , then we cannot write 0 as a sum between an element in  $F(x) \cap B(2, 2^{-1})$  and an element in  $G(x) \cap B(-2, 2^{-1})$ .

Remind that a multifunction  $F : X \rightrightarrows Y$  is said to be Lipschitz around  $\bar{x} \in \text{Dom } F$  with constant  $L > 0$  if there exists a neighborhood  $U \in \mathcal{V}(\bar{x})$  such that, for every  $x, u \in U$ ,

$$F(x) \subset F(u) + L \|x - u\| \mathbb{D}_Y. \quad (4.6)$$

Of course, this property is (much) stronger than the Lipschitz-like property, having the great advantage to be stable at summation. More precisely, if  $F, G$  are two multifunctions which are Lipschitz around some point  $\bar{x} \in \text{Dom } F \cap \text{Dom } G$ , then  $F + G$  is Lipschitz around  $\bar{x}$ .

The next, more elaborated example, shows that the Lipschitz property of both multifunctions does not ensure the sum-stable property.

**Example 4.5** Let  $F : \mathbb{R} \rightrightarrows \mathbb{R}$  be given by  $F(x) := [1, +\infty[ \cap \mathbb{Q}$  for every  $x \in \mathbb{R}$  and  $G : \mathbb{R} \rightrightarrows \mathbb{R}$  given by

$$G(x) := \begin{cases} ]-\infty, -1], & \text{if } x \in \mathbb{R} \setminus \{1 - \frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\}\} \\ ]-\infty, -1] \cap \mathbb{Q} \cup \{-2 + \frac{\sqrt{2}}{n}\}, & \text{if } x = 1 - \frac{1}{n}, n \in \mathbb{N} \setminus \{0\}. \end{cases}$$

It is easy to verify that both  $F$  and  $G$  are Lipschitz around  $\bar{x} := 1$ , because, for example, if we add to the set  $]-\infty, -1] \cap \mathbb{Q} \cup \{-2 + \frac{\sqrt{2}}{n}\}$  any ball, then we cover all the interval  $]-\infty, -1]$ . Take now  $\bar{y} := 1$  and  $\bar{z} := -1$ , fix  $\varepsilon := 2^{-1}$  and pick arbitrary  $\delta > 0$ . Then choose  $x_n := 1 - \frac{1}{n}$ ,  $n \in \mathbb{N} \setminus \{0\}$  and observe that  $x_n \in ]1 - \delta, 1 + \delta[$  and

$$w_n := \frac{\sqrt{2}}{n} \in (F + G)(x_n) \cap B(0, \delta)$$

for any  $n$  sufficiently large. But  $w_n$  can be obtained only by the sum between  $2 \in F(x_n)$  and  $-2 + \frac{\sqrt{2}}{n} \in G(x_n)$  and both these values are not in the balls  $B(\bar{y}, 2^{-1})$  and  $B(\bar{z}, 2^{-1})$  respectively. Therefore,  $(F, G)$  is not sum-stable at  $(\bar{x}, \bar{y}, \bar{z}) = (1, 1, -1)$ .

Next proposition indicates a first general situation where the local-sum stability holds. Recall that a function  $f : X \rightarrow Y$  is calm at  $\bar{x}$  if there exist  $\alpha, l > 0$  such that, for every  $x \in B(\bar{x}, \alpha)$ ,

$$\|f(x) - f(\bar{x})\| \leq l \|x - \bar{x}\|.$$

**Proposition 4.6** Let  $f : X \rightarrow Y$  be a function,  $G : X \rightrightarrows Y$  be a multifunction and  $\bar{x} \in X$  such that  $0 \in f(\bar{x}) + G(\bar{x})$ . If  $f$  is calm at  $\bar{x}$ , then  $(f, G)$  is locally sum-stable around  $(\bar{x}, f(\bar{x}), -f(\bar{x}))$ .

**Proof.** Suppose that the constants from the calmness property of  $f$  are the same as above, take arbitrarily  $\varepsilon > 0$  and choose  $\delta \in ]0, \min\{2^{-1}\varepsilon, (2l)^{-1}\varepsilon, \alpha\}[$ . Pick now  $x \in B(\bar{x}, \delta)$  and  $w \in (f + G)(x) \cap B(0, \delta)$ . Then  $w - f(x) \in G(x)$ . Moreover,  $\|f(x) - f(\bar{x})\| \leq l \|x - \bar{x}\| < l\delta < 2^{-1}\varepsilon$ , so  $f(x) \in B(f(\bar{x}), \varepsilon)$ . Also,  $w - f(x) \in B(0, \delta) + B(-f(\bar{x}), 2^{-1}\varepsilon) \subset B(-f(\bar{x}), \varepsilon)$ . The proof is now complete.  $\square$

The next lemma is the main motivation for introducing the sum-stable property.

**Lemma 4.7** Let  $F : X \rightrightarrows Y$ ,  $G : X \rightrightarrows Y$  be two multifunctions. Suppose that  $F$  is Lipschitz-like around  $(\bar{x}, \bar{y}) \in \text{Gr } F$ , that  $G$  is Lipschitz-like around  $(\bar{x}, \bar{z}) \in \text{Gr } G$  and that  $(F, G)$  is locally sum-stable around  $(\bar{x}, \bar{y}, \bar{z})$ . Then the multifunction  $F + G$  is Lipschitz-like around  $(\bar{x}, \bar{y} + \bar{z})$ . Moreover, the following relation holds true

$$\text{lip}(F + G)(\bar{x}, \bar{y} + \bar{z}) \leq \text{lip } F(\bar{x}, \bar{y}) + \text{lip } G(\bar{x}, \bar{z}). \quad (4.7)$$



**Proof.** Using the Lipschitz-like properties of  $F$  and  $G$ , one can find  $\alpha, l, k > 0$  such that, for every  $x, u \in B(\bar{x}, \alpha)$ ,

$$F(x) \cap B(\bar{y}, \alpha) \subset F(u) + l \|x - u\| \mathbb{D}_Y, \quad (4.8)$$

$$G(x) \cap B(\bar{z}, \alpha) \subset G(u) + k \|x - u\| \mathbb{D}_Y. \quad (4.9)$$

But using the local sum-stability for  $\varepsilon := \alpha > 0$ , we can find  $\delta \in ]0, \alpha[$  such that, for every  $x \in B(\bar{x}, \delta)$  and every  $w \in (F + G)(x) \cap B(\bar{y} + \bar{z}, \delta)$ , there exist  $y \in F(x) \cap B(\bar{y}, \alpha)$  and  $z \in G(x) \cap B(\bar{z}, \alpha)$  such that  $w = y + z$ . Consequently, using (4.8) and (4.9), for every  $u \in B(\bar{x}, \delta)$ ,  $w \in (F + G)(u) + (l + k) \|x - u\| \mathbb{D}_Y$ . The relation (4.7) follows from the fact that constants  $l$  and  $k$  can be chosen arbitrarily close to  $\text{lip } F(\bar{x}, \bar{y})$  and  $\text{lip } G(\bar{x}, \bar{z})$ , respectively.  $\square$

We would like to continue with our examples above in order to illustrate the fact that sum-stable property is essential in Lemma 4.7. Basically, we need an example of two Lipschitz-like multifunctions for which the sum is not Lipschitz-like at the reference point.

**Example 4.8** Let  $F : \mathbb{R} \rightrightarrows \mathbb{R}$  be given by

$$F(x) := \begin{cases} [1, 2], & \text{if } x \in \mathbb{R} \setminus \{1\} \\ [1, 2] \cup \{0\}, & \text{if } x = 1 \end{cases}$$

and  $G : \mathbb{R} \rightrightarrows \mathbb{R}$  given by  $G(x) := [1, 2]$  for every  $x \in \mathbb{R}$ .

As above, it is easy to see that both  $F$  and  $G$  are Lipschitz-like around  $(\bar{x}, \bar{y}) = (1, 1)$  and  $(\bar{x}, \bar{z}) = (1, 1)$ , respectively. But the multifunction  $F + G : \mathbb{R} \rightrightarrows \mathbb{R}$ , given by

$$(F + G)(x) = \begin{cases} [2, 4], & \text{if } x \in \mathbb{R} \setminus \{1\} \\ [1, 4], & \text{if } x = 1 \end{cases},$$

is not Lipschitz-like around  $(\bar{x}, \bar{y} + \bar{z}) = (1, 2)$ . Indeed, suppose by contradiction that there exists  $L > 0$  and  $\alpha \in ]0, \min\{1, L^{-1}\}[$  such that for any  $x, u \in [1 - \alpha, 1 + \alpha]$

$$(F + G)(x) \cap [2 - \alpha, 2 + \alpha] \subset (F + G)(u) + L \|x - u\| [-1, 1]. \quad (4.10)$$

Consider now  $x := 1$  and  $u := 1 - \alpha^2$  such that  $x, u \in [1 - \alpha, 1 + \alpha]$ . Clearly,

$$2 - \alpha \in (F + G)(x) \cap [2 - \alpha, 2 + \alpha].$$

Following (4.10), we should have:

$$2 - \alpha \in (F + G)(1 - \alpha^2) + L\alpha^2[-1, 1]$$

and, in particular,

$$2 - \alpha \in [2 - L\alpha^2, 4 + L\alpha^2].$$

But this requires that

$$\alpha \leq L\alpha^2,$$

which contradicts the choice of  $\alpha$ . The contradiction shows that we cannot have the Lipschitz-like property of the sum.

Observe that, in the virtue of Lemma 4.7,  $(F, G)$  cannot be locally-sum stable around  $(1, 1, 1)$ . Indeed, take  $\varepsilon \in ]0, 2^{-1}[$ . Then for every  $\delta > 0$ , choose  $n \in \mathbb{N}$  such that  $n > \max\{\delta, 1\}$ . Taking now  $x_\delta := 1 \in ]1 - \delta, 1 + \delta[$  and  $w_\delta := 2 - n^{-1}\delta \in (F + G)(x_\delta) \cap ]2 - \delta, 2 + \delta[$ , one can easily see that, for every  $y \in F(x_\delta) \cap ]1 - \varepsilon, 1 + \varepsilon[ = [1, 1 + \varepsilon[$  and every  $z \in G(x_\delta) \cap ]1 - \varepsilon, 1 + \varepsilon[ = [1, 1 + \varepsilon[$ ,  $w_\delta < y + z$ .

Next, we adapt the definition of local-sum stability to the parametric case, in order to use this notion in the general context of variational systems.

**Definition 4.9** Let  $F : X \times P \rightrightarrows Y$ ,  $G : X \rightrightarrows Y$  be two multifunctions and  $(\bar{x}, \bar{p}, \bar{y}, \bar{z}) \in X \times P \times Y \times Y$  such that  $\bar{y} \in F(\bar{x}, \bar{p})$ ,  $\bar{z} \in G(\bar{x})$ . We say that the multifunction  $(F, G)$  is locally sum-stable around  $(\bar{x}, \bar{p}, \bar{y}, \bar{z})$  if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that, for every  $(x, p) \in B(\bar{x}, \delta) \times B(\bar{p}, \delta)$  and every  $w \in (F_p + G)(x) \cap B(\bar{y} + \bar{z}, \delta)$ , there exist  $y \in F_p(x) \cap B(\bar{y}, \varepsilon)$  and  $z \in G(x) \cap B(\bar{z}, \varepsilon)$  such that  $w = y + z$ .

Similarly to Proposition 4.6, one can easily prove the next (adapted) result.

**Proposition 4.10** Let  $f : X \times P \rightarrow Y$  be a function,  $G : X \rightrightarrows Y$  be a multifunction and  $(\bar{x}, \bar{p}) \in X \times P$  such that  $0 \in f(\bar{x}, \bar{p}) + G(\bar{x})$ . If  $f$  is calm at  $(\bar{x}, \bar{p})$ , then  $(f, G)$  is locally sum-stable around  $(\bar{x}, \bar{p}, f(\bar{x}, \bar{p}), -f(\bar{x}, \bar{p}))$ .

Also, Lemma 4.7 has the following variant in the parametric case.

**Lemma 4.11** Let  $F : X \times P \rightrightarrows Y$ ,  $G : X \rightrightarrows Y$  be two multifunctions. Suppose that  $F$  is Lipschitz-like with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), \bar{y}) \in \text{Gr } F$ , that  $G$  is Lipschitz-like around  $(\bar{x}, \bar{z}) \in \text{Gr } G$  and that  $(F, G)$  is locally sum-stable around  $(\bar{x}, \bar{p}, \bar{y}, \bar{z})$ . Then the multifunction  $H : X \times P \rightrightarrows Y$  given by  $H(x, p) := F(x, p) + G(x)$  is Lipschitz-like with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), \bar{y} + \bar{z})$ . Moreover, the following relation holds true

$$\widehat{\text{lip}}_x H((\bar{x}, \bar{p}), \bar{y} + \bar{z}) \leq \widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y}) + \text{lip } G(\bar{x}, \bar{z}). \quad (4.11)$$

The following result deduces the metric regularity of  $S$  under appropriate assumptions on the multifunctions  $F$  and  $G$ , which arrive naturally from Theorem 3.6. Namely, part of these assumptions are stated in order to ensure the Lipschitz-like property of the sum multifunction  $H$  with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), 0)$ , with (4.11) satisfied. Using (3.10), we expect to have that  $\text{reg } S(\bar{p}, \bar{x}) \leq c^{-1} \widehat{\text{lip}}_x H((\bar{x}, \bar{p}), 0)$ , where  $c > 0$  is the rate of linear openness with respect to  $p$  of  $H$ , but is easy to see that  $c$  must be  $\widehat{\text{lop}}_p F((\bar{x}, \bar{p}), \bar{y}) = (\widehat{\text{reg}}_p F((\bar{x}, \bar{p}), \bar{y}))^{-1}$ . Therefore, (4.12) below holds in a natural way.

**Theorem 4.12** Let  $X, Y, P$  be Banach spaces,  $F : X \times P \rightrightarrows Y$ ,  $G : X \rightrightarrows Y$  be two set-valued maps and  $(\bar{x}, \bar{p}, \bar{y}) \in X \times P \times Y$  such that  $\bar{y} \in F(\bar{x}, \bar{p})$  and  $-\bar{y} \in G(\bar{x})$ . Suppose that the following assumptions are satisfied:

- (i)  $(F, G)$  is locally sum-stable around  $(\bar{x}, \bar{p}, \bar{y}, -\bar{y})$ ;
- (ii)  $F$  is Lipschitz-like with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), \bar{y})$ ;
- (iii)  $F$  is metrically regular with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), \bar{y})$ ;
- (iv)  $G$  is Lipschitz-like around  $(\bar{x}, -\bar{y})$ .

Then  $S$  is metrically regular around  $(\bar{p}, \bar{x})$ . Moreover, the next relation holds

$$\text{reg } S(\bar{p}, \bar{x}) \leq \widehat{\text{reg}}_p F((\bar{x}, \bar{p}), \bar{y}) \cdot [\widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y}) + \text{lip } G(\bar{x}, -\bar{y})]. \quad (4.12)$$

**Proof.** Define  $H : X \times P \rightrightarrows Y$  by

$$H(x, p) := F(x, p) + G(x). \quad (4.13)$$

Using Lemma 4.11, we know that  $H$  is Lipschitz-like with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), 0)$  and the relation (4.11) holds for  $\bar{z} := -\bar{y}$ .

Using now Proposition 2.2, (iii) is equivalent to the fact that  $F$  is open at linear rate with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), \bar{y})$  and  $\widehat{\text{lop}}_p F((\bar{x}, \bar{p}), \bar{y}) = (\widehat{\text{reg}}_p F((\bar{x}, \bar{p}), \bar{y}))^{-1}$ . Consequently, there exist  $\varepsilon, L > 0$  such that, for every  $x \in B(\bar{x}, \varepsilon)$ , every  $(p, y) \in \text{Gr } F_x \cap [B(\bar{p}, \varepsilon) \times B(\bar{y}, \varepsilon)]$  and every  $\rho \in ]0, \varepsilon[$ , one has

$$B(y, L\rho) \subset F_x(B(p, \rho)).$$

Using now (i), there exists  $\delta \in ]0, \varepsilon[$  such that, for every  $(x, p) \in B(\bar{x}, \delta) \times B(\bar{p}, \delta)$  and every  $w \in (F_p + G)(x) \cap B(0, \delta)$ , there exist  $y \in F_p(x) \cap B(\bar{y}, \varepsilon)$  and  $z \in G(x) \cap B(-\bar{y}, \varepsilon)$  such that  $w = y + z$ .

Take now arbitrary  $x \in B(\bar{x}, \delta)$ ,  $(p, w) \in \text{Gr } H_x \cap [B(\bar{p}, \delta) \times B(0, \delta)]$  and  $\rho \in ]0, \varepsilon[$ . Then  $w \in H(x, p) \cap B(0, \delta)$ , so  $w$  can be written as  $y + z$ , with  $(p, y) \in \text{Gr } F_x \cap [B(\bar{p}, \varepsilon) \times B(\bar{y}, \varepsilon)]$  and  $z \in G(x) \cap B(-\bar{y}, \varepsilon)$ , whence

$$B(w, L\rho) = z + B(y, L\rho) \subset G(x) + F(x, B(p, \rho)) = H_x(B(p, \rho)).$$

Adding the fact that  $L$  can be chosen arbitrarily close to  $\widehat{\text{lop}}_p F((\bar{x}, \bar{p}), \bar{y})$ , we obtain that  $H$  is open at linear rate with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), 0)$  and  $\widehat{\text{lop}}_p H((\bar{x}, \bar{p}), 0) = (\widehat{\text{reg}}_p F((\bar{x}, \bar{p}), \bar{y}))^{-1}$ .

Now the result easily follows from the second item of Theorem 3.6.  $\square$

Theorem 4.12 is a generalization of [1, Theorem 3.3 (i)], concerning the direct implication.

Notice that in [15, Theorems 3.2, 3.3] some related results are obtained as well. However, Uderzo's approach, which works on metric spaces, concerns global openness results and involves a parametric function instead of our multifunction  $F$ , and  $G$  is also taken in parametric form.

On the other hand, the next result uses the ideas of the proof of Theorem 3.6 and, essentially, the estimations from Theorem 3.3. For several technical reasons it is not possible to give a direct and easy proof based on the main results, but, nevertheless, the proof uses the very same ideas and arguments arranged in the specific context of this result.

Let us emphasize that, once again, the Lipschitz modulus of  $S$  has a form which can be easily developed from the previous facts. Namely, one can expect that  $\text{lip } S(\bar{p}, \bar{x}) \leq c^{-1} \widehat{\text{lip}}_p H((\bar{x}, \bar{p}), 0) = c^{-1} \widehat{\text{lop}}_p F((\bar{x}, \bar{p}), \bar{y})$ , where  $c > 0$  must be the rate of linear openness of  $H$  with respect to  $x$ . One can observe that for every  $p$  in a neighborhood of  $\bar{p}$ , and for appropriate  $x, y$ ,  $\widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y})$  seems to be very close to  $\text{lip}(F_p)(x, y) = \text{lip}(-F_p)(x, -y) = (\text{lop}(-F_p)^{-1}(-y, x))^{-1}$ . Also,  $\text{reg } G(\bar{x}, -\bar{y}) = (\text{lop } G(\bar{x}, -\bar{y}))^{-1}$ . Because we know additionally that  $\widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y}) \cdot \text{reg } G(\bar{x}, -\bar{y}) < 1$ , then one can expect that  $G - (-F_p)$  might be open at the linear rate  $c = \text{lop } G(\bar{x}, -\bar{y}) - \widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y}) = (\text{reg } G(\bar{x}, -\bar{y}))^{-1} - \widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y})$  at points close to  $((\bar{x}, \bar{p}), 0)$ . Whence, once again, one can have an intuitive approach in getting (4.14) below.

**Theorem 4.13** *Let  $X, Y, P$  be Banach spaces,  $F : X \times P \rightrightarrows Y$ ,  $G : X \rightrightarrows Y$  be two set-valued maps and  $(\bar{x}, \bar{p}, \bar{y}) \in X \times P \times Y$  such that  $\bar{y} \in F(\bar{x}, \bar{p})$  and  $-\bar{y} \in G(\bar{x})$ . Suppose that the following assumptions are satisfied:*

- (i)  $(F, G)$  is locally sum-stable with respect to  $x$  uniformly in  $p$  around  $(\bar{x}, \bar{p}, \bar{y}, -\bar{y})$ ;
- (ii) for every  $p$  in a neighborhood of  $\bar{p}$ ,  $\text{Gr } F_p$  is closed;
- (iii)  $\text{Gr } G$  is closed;
- (iv)  $F$  is Lipschitz-like around  $((\bar{x}, \bar{p}), \bar{y})$ ;

(v)  $G$  is metrically regular around  $(\bar{x}, -\bar{y})$ ;

(vi)  $\widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y}) \cdot \text{reg } G(\bar{x}, -\bar{y}) < 1$ .

Then  $S$  is Lipschitz-like around  $(\bar{p}, \bar{x})$ . Moreover, the next relation is satisfied

$$\text{lip } S(\bar{p}, \bar{x}) \leq \frac{\text{reg } G(\bar{x}, -\bar{y}) \cdot \widehat{\text{lip}}_p F((\bar{x}, \bar{p}), \bar{y})}{1 - \widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y}) \cdot \text{reg } G(\bar{x}, -\bar{y})}. \quad (4.14)$$

**Proof.** Take  $m > \text{reg } G(\bar{x}, -\bar{y})$  and  $l > \widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y})$  such that  $m \cdot l < 1$ .

We intend to prove that there exist  $\tau, t, \gamma > 0$  such that, for every  $(x, p) \in B(\bar{x}, \tau) \times B(\bar{p}, t)$ ,

$$d(x, S(p)) \leq m(1 - lm)^{-1} d(0, [F(x, p) + G(x)] \cap B(0, \gamma)). \quad (4.15)$$

Using the assumption (iv), one can find  $r_1, t_1 > 0$  such that, for every  $p \in B(\bar{p}, t_1)$ , and every  $x, x' \in B(\bar{x}, r_1)$ ,

$$F_p(x) \cap B(\bar{y}, s_1) \subset F_p(x') + l \|x - x'\| \mathbb{D}_Y. \quad (4.16)$$

But this shows, as one can see next, that for every  $p \in B(\bar{p}, t_1)$ ,  $F_p$  is  $l$ -pseudocalm at every  $(x, y) \in \text{Gr } F_p \cap [B(\bar{x}, 2^{-1}r_1) \times B(\bar{y}, s_1)]$ . Indeed, take  $x \in B(\bar{x}, 2^{-1}r_1)$  and  $y \in F_p(x) \cap B(\bar{y}, s_1)$ . Then for every  $x' \in B(x, 2^{-1}r_1)$ , we have from (4.16) that there exists  $y' \in F_p(x')$  such that

$$d(y, F_p(x')) \leq \|y - y'\| \leq l \|x - x'\|,$$

which proves the desired assertion. Hence, we conclude in view of Proposition 2.4 that for every  $p \in B(\bar{p}, t_1)$ ,  $F_p^{-1}$  is  $l^{-1}$ -open at every  $(y, x) \in \text{Gr } F_p^{-1} \cap [B(\bar{y}, s_1) \times B(\bar{x}, 2^{-1}r_1)]$ .

Also, from (v), we know that there exist  $r_2, s_2 > 0$  such that, for every  $(u, v) \in \text{Gr } G \cap [B(\bar{x}, r_2) \times B(-\bar{y}, s_2)]$ ,  $G$  is metrically hemiregular at  $(u, v)$  with constant  $m$ , whence is open at linear rate  $m^{-1}$  at  $(u, v)$ .

Use now the property from (i) for  $\min\{2^{-1}s_1, 2^{-1}s_2\}$  instead of  $\varepsilon$  and find  $\delta$  such that the assertion from Definition 4.9 is true.

Take now  $\rho \in ]0, \min\{(m^{-1} - l)^{-1}\delta, 4^{-1}r_1, 2^{-1}r_2, 2^{-1}l^{-1}s_1, 2^{-1}ms_2\}[$  and define  $\gamma := (m^{-1} - l)\rho$ . Suppose that  $B(\bar{p}, t_2)$  is the neighborhood which appears in (ii), denote by  $t := \min\{t_1, t_2\}$ ,  $\tau := \min\{4^{-1}r_1, 2^{-1}r_2\}$ , and take  $(x, p) \in B(\bar{x}, \tau) \times B(\bar{p}, t)$ .

If  $[F(x, p) + G(x)] \cap B(0, \gamma) = \emptyset$  or  $0 \in [F(x, p) + G(x)] \cap B(0, \gamma)$ , then (4.15) trivially holds. Suppose that  $0 \notin [F(x, p) + G(x)] \cap B(0, \gamma)$ . Then, for every  $\varepsilon > 0$ , one can find  $w_\varepsilon \in [F(x, p) + G(x)] \cap B(0, \gamma)$  such that

$$\|w_\varepsilon\| < d(0, [F(x, p) + G(x)] \cap B(0, \gamma)) + \varepsilon. \quad (4.17)$$

Obviously,  $d(0, [F(x, p) + G(x)] \cap B(0, \gamma)) < (m^{-1} - l)\rho$ , so for  $\varepsilon > 0$  sufficiently small,  $d(0, [F(x, p) + G(x)] \cap B(0, \gamma)) + \varepsilon < (m^{-1} - l)\rho$ . Consequently, we get from (4.17) that

$$0 \in B(w_\varepsilon, d(0, [F(x, p) + G(x)] \cap B(0, \gamma)) + \varepsilon) \subset B(w_\varepsilon, (m^{-1} - l)\rho) = B(w_\varepsilon, \delta), \quad (4.18)$$

Applying (i), one can find  $y_\varepsilon \in F(p, x) \cap B(\bar{y}, 2^{-1}s_1)$  and  $z_\varepsilon \in G(x) \cap B(-\bar{y}, 2^{-1}s_2)$  such that  $w_\varepsilon = y_\varepsilon + z_\varepsilon$ . Whence,  $B(y_\varepsilon, 2^{-1}s_1) \subset B(\bar{y}, s_1)$  and  $B(z_\varepsilon, 2^{-1}s_2) \subset B(-\bar{y}, s_2)$ .

Denote now  $r'_1 := 4^{-1}r_1$ ,  $s'_1 := 2^{-1}s_1$ ,  $r'_2 := 2^{-1}r_2$ ,  $s'_2 := 2^{-1}s_2$ . Summarizing,  $F_p^{-1}$  is  $l^{-1}$ -open at every  $(y', x') \in \text{Gr } F_p^{-1} \cap [B(y_\varepsilon, s'_1) \times B(x, r'_1)]$ ,  $G$  is  $m^{-1}$ -open at every  $(u', v') \in \text{Gr } G \cap [B(x, r'_2) \times B(z_\varepsilon, s'_2)]$  and  $l^{-1}m^{-1} > 1$ . We can apply then Theorem 3.3 for  $-G$ ,  $F_p^{-1}$ ,  $(x, -z_\varepsilon) \in$

$\text{Gr}(-G)$ ,  $(y_\varepsilon, x) \in \text{Gr } F_p^{-1}$  and  $\rho_0 := (m^{-1} - l)^{-1}(d(0, [F(x, p) + G(x)] \cap B(0, \gamma)) + \varepsilon) < \rho < \min\{r'_1, r'_2, l^{-1}s'_1, ms'_2\}$  to obtain that

$$B(y_\varepsilon + z_\varepsilon, d(0, [F(x, p) + G(x)] \cap B(0, \delta)) + \varepsilon) \subset (F_p + G)(B(x, \rho_0)).$$

Using (4.18), we obtain that  $0 \in (F_p + G)(B(x, \rho_0))$ , so there exists  $\tilde{x} \in B(x, \rho_0)$  such that  $0 \in F(\tilde{x}, p) + G(\tilde{x})$  or, equivalently,  $\tilde{x} \in S(p)$ . Hence

$$d(x, S(p)) \leq \|x - \tilde{x}\| < \rho_0 = (m^{-1} - l)^{-1}d(0, [F(x, p) + G(x)] \cap B(0, \delta)) + \varepsilon.$$

Making  $\varepsilon \rightarrow 0$ , we obtain (4.15).

For the final step of the proof, observe that because  $F$  is Lipschitz-like with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), \bar{y})$ , there exist  $\alpha, k > 0$  such that for every  $x \in B(\bar{x}, \alpha)$ , every  $p_1, p_2 \in B(\bar{p}, \alpha)$ , one has

$$F_x(p_1) \cap B(\bar{y}, \alpha) \subset F_x(p_2) + k \|p_1 - p_2\| \mathbb{D}_Y. \quad (4.19)$$

Use now (i) for  $\alpha$  instead of  $\varepsilon$  and find  $\delta' \in ]0, \alpha[$  such that the assertion from Definition 4.9 is true. Take now arbitrary  $x \in B(\bar{x}, \delta')$ ,  $p_1, p_2 \in B(\bar{p}, \delta')$  and  $w \in H_x(p_1) \cap B(0, \delta')$ , where  $H$  is defined by (4.13). Then there exist  $y \in F_x(p_1) \cap B(\bar{y}, \alpha)$  and  $z \in G(x) \cap B(-\bar{y}, \alpha)$  such that  $w = y + z$ . Using now (4.19), we have that

$$\begin{aligned} w &= y + z \in F(x, p_2) + k \|p_1 - p_2\| \mathbb{D}_Y + G(x) \\ &= H_x(p_2) + k \|p_1 - p_2\| \mathbb{D}_Y. \end{aligned}$$

Consequently,  $H$  is Lipschitz-like with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), 0)$ . Moreover, because  $k$  can be chosen arbitrarily close to  $\widehat{\text{lip}}_p F((\bar{x}, \bar{p}), \bar{y})$ , we have that  $\widehat{\text{lip}}_p F((\bar{x}, \bar{p}), \bar{y}) = \widehat{\text{lip}}_p H((\bar{x}, \bar{p}), 0)$ . Then one can proceed as in the proof of the final part of Theorem 3.6 (i) to conclude the proof.  $\square$

The result would follow using just the local closedness of the graphs of  $F_p$  and  $G$  for  $p$  close to  $\bar{p}$  around appropriate points, but we preferred the actual formulation in order to avoid a more complicated assertion. Theorem 4.13 generalizes [1, Theorem 5.1 (ii)].

Note that, using Proposition 4.1, one could obtain on the basis of Artacho and Mordukhovich tools in [1] the converse links between the metric regularity/Lipschitz-like property of solution map  $S$  and the corresponding properties of the field map  $G$ .

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